MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY,

CONTAINING

PAPERS, ABSTRACTS OF PAPERS, AND

REPORTS OF THE PROCEEDINGS

OF THE SOCIETY

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MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY.

VOL. LXXX. NOVEMBER 14, 1919. No. 1

Professor A. FOWLER, F.R.S., PRESIDENT, in the Chair.

Charles Galton Darwin, M.A., M.C., Fellow of Christ's College, Cambridge;

Edwin Powell Hubble, Ph.D., Mount Wilson Observatory, Pasadena, California, U.S.A.; and

Archibald William Tindall Lucas, Berton Hatch, Horsell, Woking,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Jules Baillaud, Docteur-ès-Sciences, Astronome Adjoint à l'Observatoire de Paris, 148 Boulevard du Montparnasse, Paris, XIV^{e} (proposed by Sir Frank Dyson);

William Wood Burkett, 15 Vanbrugh Fields, Blackheath, S.E. (proposed by Sir Frank Dyson);

Fred Clements, M.Inst.C.E., M.I.Mech.E., The Quintas, Spital, Chesterfield (proposed by E. Howarth);

Ralph E. De Lury, M.A., Ph.D., Astronomer, Dominion Observatory, Ottawa, Canada (proposed by Otto Klotz);

Wilfred Hall, M.A., Hepple Woodside, Rothbury, Northumberland (proposed by Harold Thomson);

Frank Jeffries, 120 Malpas Road, Brockley, S.E. 4 (proposed by Sir Frank Dyson);

Ernest Marsden, D.Sc., M.C., Professor of Physics, Victoria University College, Wellington, New Zealand (proposed by C. E. Adams);

Lord Rayleigh, 69 Cadogan Square, S.W. 1 (proposed by Alfred Fowler);

Carl Sanders, Planter, Matuba, Cabinda, Portuguese West Africa (proposed by W. de Sitter);

James Francis Schofield Shepherd, M.A., 9 Park Road, Halifax (proposed by the Rev. Charles Whyte);

The Rev. Samuel Paul Shipley, The Manse, Oakham (proposed by the Rev. J. T. W. Claridge);

David Williams, B.Sc.(Lond.), Professor of Mathematics, Rhodes University College, Grahamstown, South Africa (proposed by W. H. Finlay); and

Lieut.-Col. H. St. J. L. Winterbotham, C.M.G., D.S.O., R.E., Ordnance Survey, Southampton (proposed by Col. Sir Charles Close).

One hundred and fifty presents were announced as having been received since the last Meeting, including, amongst others:—

J. H. Jeans, Problems of Cosmogony and Stellar Dynamics, presented by the author; and

Tycho Brahe, Opera Omnia, edidit J. L. E. Dreyer, Tomus III., Tomus IV., Fasciculus I., presented by Dr. Dreyer.

Photographic Reproduction of the Franklin-Adams Chart of the Sky.

The original issue of these Charts having been exhausted six years ago, it is proposed to prepare a second edition of the reproductions.

Owing to the increased cost of materials, the price of the set of 206 charts, in three cases, has been raised to £20.

Applications for sets should be made at once. It is expected that the first sets will be ready in about a year's time. Further particulars are given on the enclosed slip.

The portrait of Sir David Gill by George Henry, belonging to the Royal Society, has been photographed in permanent bromide: size of print 11 in. × $8\frac{3}{4}$ in., mounted on plate, sunk mount $15\frac{1}{4}$ in. × $12\frac{1}{4}$ in. for framing, and orders for copies will be received by the Assistant Secretary. The price is 6s., including packing and postage. Orders should be accompanied by remittance.

The Region of the Sky between R.A. 3^h *and* 5^h 30^m *and N. Dec.* 20° *to* 35°. By Sir F. W. Dyson and P. J. Melotte. (Plate 1.)

1. In the introduction to the Greenwich Catalogue for 1910, which contains all stars down to $9^m{\cdot}0$ between the limits +24° and 32° N. Dec., a table is formed giving the number of stars in each hour. These numbers are as follows:—

h h		h h		h h		h h	
0–1	485	6–7	731	12–13	352	18–19	692
1–2	424	7–8	530	13–14	315	19–20	976
2–3	406	8–9	429	14–15	345	20–21	852
3–4	399	9–10	377	15–16	358	21–22	670
4–5	339	10–11	364	16–17	378	22–23	575
5–6	641	11–12	302	17–18	542	23–0	511

2. The regular rise and fall of the numbers, according to galactic latitude, is markedly interrupted at 4^h to 5^h. It seemed worth while to look more closely into this peculiarity, and a careful examination was made of the Franklin-Adams charts. It was at once seen from the prints that between 3^h and 5^h 30^m there were areas of considerable extent almost entirely devoid of stars. In the whole zone 24° – 32° N. Dec. there are only four other very small areas at all comparable with these. The outline is so clearly defined on the prints of the Franklin-Adams chart that it was easy with a pencil to trace out the sparse areas. The dark areas on the accompanying diagram are those originally traced out in this manner by Mr. Melotte.

3. After this had been done, it was decided to make complete counts of the original Franklin-Adams chart plates covering the whole region. Generally speaking, at each intersection of the réseau lines, which are 1° apart, an area of 400 square minutes was counted. In and immediately around the sparse areas the counts were made over larger areas. The figures given on the accompanying diagram are the number of stars per 100 square minutes down to a magnitude estimated to be $15{\cdot}8$ photographic.

The greater part of the area considered is covered by three Franklin-Adams chart plates at centres R.A. 3^h, 4^h, and 5^h, and Dec. +30°. South of Dec. +22° the counts were made on Franklin-Adams chart plates whose centres are at Dec. +15°. Use was made of the overlap which exists between the plates to obtain a uniform scale of limiting magnitude. The accordance of the different plates was quite satisfactory.

4. It was necessary to correct these counts for the falling off in magnitude (reaching to approximately $0^m{\cdot}8$) of the Franklin-Adams plates from the centre towards the edge. The following

table obtained from a consideration of 43 plates in zones $+75°$, $60°$, and $45°$ shows the loss in photographic magnitude, from the centre towards the edge, for twenty-five regions of $3° \times 3°$ square.

m	m	m	m	m
0·79	0·73	0·69	0·73	0·79
0·73	0·42	0·24	0·42	0·73
0·69	0·24	0·00	0·24	0·69
0·73	0·42	0·24	0·42	0·73
0·79	0·73	0·69	0·73	0·79

This table was obtained by Dr. Chapman and Mr. Melotte from a consideration of the numbers. It was verified through the intermediary of photometry plates taken with the 26-inch refractor, by which means the central area of the chart at R.A. 5^h Dec. $+30°$ and an area $5\frac{1}{2}°$ from the centre of the same chart were definitely compared with Pickering's polar standards down to $14^m{\cdot}2$. At this distance a falling off of $0^m{\cdot}66$ was shown for the faint stars, agreeing satisfactorily with the table. In this way it was possible to reach a fairly uniform limit of magnitude over the whole area shown on the diagram. The limiting magnitude is not accurately fixed, but may be taken at about $15^m{\cdot}8$, and should be within $0^m{\cdot}25$ of this figure.

After the counts had been set out, contour lines were drawn, marking the areas where the numbers are greater than 20 per 100 square minutes, and greater than 40. Examination of the diagram shows the extent and suddenness of the falling off of numbers inside the shaded areas. In the area at $5^h\ 20^m + 25°$ the star density is 10·5, while in the surrounding regions it is 28 or 2·7 times as great. In the large central area the greater part has a density of about 3, while in the surrounding region it is 19. For the whole area the density is 1 : 5 of the surrounding region. In the area about $3^h\ 30^m + 30°$ the density is about 6, while outside it is 16 or a ratio of 1 to 2·7.

If these figures are to be interpreted as arising from absorbing matter, we may say that over the central area at least two magnitudes are absorbed, and over the other two areas more than one magnitude.

5. A complete count was made over an area of 3 square degrees in the part of smallest density, giving the numbers :—

$0^m{\cdot}0-10^m{\cdot}0$	2
$10^m{\cdot}0-11^m{\cdot}0$	4
$11^m{\cdot}0-12^m{\cdot}0$	8
$12^m{\cdot}0-13^m{\cdot}0$	19
$13^m{\cdot}0-14^m{\cdot}0$	23
$14^m\ 0-15^m{\cdot}0$	50

The number of stars per square degree in these three square degrees compared with

(1) Count made on plate taken with the 26-inch refractor with centre at 5^h 0^m +30°;

(2) Count made on plate taken with 26-inch refractor of Kapteyn Area 49 (R.A. 5^h 22^m Dec. +29° 43′)

is shown in the following table:—

Mag.	4^h 30^m +26° Gal. Lat. −13°.	5^h 0^m +30° Gal. Lat. −6°.	5^h 22^m +29° 43′ Gal. Lat. −3°.
brighter than 10·0	0·7	0	0
,, ,, 11·0	2·0	32	27
,, ,, 12·0	4·7	56	58
,, ,, 13·0	11·0	142	146
,, ,, 14·0	18·7	277	302
,, ,, 15·0	35·3		

Thus up to $14^m{\cdot}0$ the proportion of stars appears to be approximately constant and in the ratio of 1 : 14, but the number of stars brighter than $11^m{\cdot}0$ is so small that the table can only be considered as applying from $11^m{\cdot}0$ to $14^m{\cdot}0$.

From $11^m{\cdot}0$–$12^m{\cdot}0$ the ratio is 1 : 10;
,, $12^m{\cdot}0$–$13^m{\cdot}0$,, 1 : 14;
,, $13^m{\cdot}0$–$14^m{\cdot}0$,, 1 : 19.

The fall in numbers corresponds

for $11^m{\cdot}0$–$12^m{\cdot}0$ to a loss of 2·6 magnitudes;
,, $12^m{\cdot}0$–$13^m{\cdot}0$,, 3·3 ,,
,, $13^m{\cdot}0$–$14^m{\cdot}0$,, 3·9 ,,

These figures should not be pressed unduly, but seem to show that the fainter stars are affected to a greater extent.

6. We may now examine the corresponding region in the B.D. The outlines of the three dark areas were drawn in pencil on a B.D. chart. The following table gives a comparison of the star density in the sparse areas with the star density in surrounding regions:—

	Central Area, 4^h 30^m+26°.	Area at 5^h 20^m+25°.	Area at 3^h 20^m+31°.
Total number of B.D. stars in area.	60	65	160
Area	23 sq. deg.	9·4 sq. deg.	18·7 sq. deg.
Number per sq. degree .	2·6	6·9	8·5
Number per sq. degree in surrounding region.	12	21	12
Ratio	1 : 5	1 : 3	1 : 1·5

Thus to the limit of the B.D. there is, in the areas at 4^h 30^m and 5^h 20^m, a falling off in the same ratio as for the fainter stars counted on the Franklin-Adams chart. In the area at 3^h 20^m,

where the obscuration is not nearly so marked, it appears to be more for stars beyond the B.D. limit. The limit of the B.D. in this region may be considered as approximately $11^m{\cdot}0$ photographic.

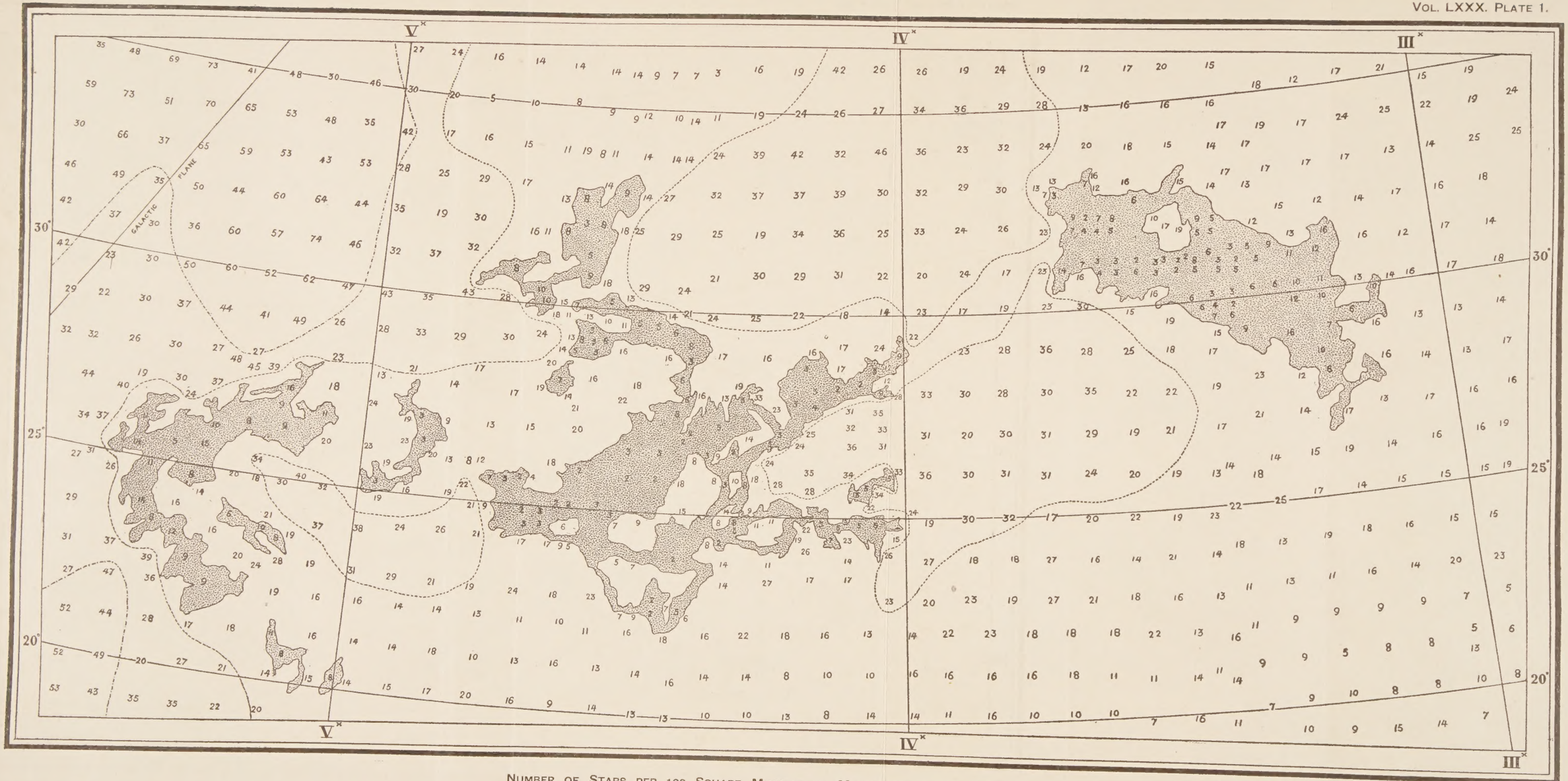
7. Considering now the brighter stars, we find in the area at R.A. 4^h 30^m Dec. +26° (23 square degrees) the following stars brighter than $9^m{\cdot}5$ on the B.D. scale:—

No. in B.D.	Mag. (H. C.).	Spect.	Proper Motion. R.A. s·0001.	Proper Motion. Dec. ″·001.	No. in B.D.	Mag. (H. C.).
°	m				°	
24 674	6·27	A_3	+ 7	+ 2	25 736	9·4
27 661	6·61	A_0	+15	− 34	22 713	9·4
27 660	7·76	A_0	−21	+ 13	29 724	9·5
30 704	8·06	F_5	+33	− 60	27 653	9·5
25 720	7·61	K_2	+21	− 12	26 728	9·5
24 689	8·0	F_2	+23	− 25	24 675	9·5
24 649	8·1	...	...	...	28 659	9·8
25 723	8·4	B_9	− 7	− 31	28 681	9·8
25 725	8·6	K_5	−18	+ 17	24 692	9·8
26 724	8·8	...	+12	− 45	25 713	10·0
27 651	8·8	K_5	+12	− 36	27 652	10·2
24 658	9·0	A	−35	− 14	25 687	10·2
29 721	9·1	...	−35	−153	24 676	10·2
29 730	9·4	...	−43	− 1	24 690	10·2
29 665	9·4	...	+ 1	+ 5	23 716	10·2
28 630	9·4	...	+37	+ 10		
28 679	9·4	...	−22	− 8		

Taking first the number of the stars: the density of stars to $9^m{\cdot}0$ as compared with neighbouring regions and other regions of the same galactic latitude is in the proportion of about 1 : 5. Thus, taking this area as a whole (and not merely the least dense part), we find the number of stars is about one-fifth of the normal number, whether we go down to magnitude $9^m{\cdot}0$, $11^m{\cdot}0$, or $14^m{\cdot}0$ (§§ 4 and 6).

This would seem to indicate that if the small density is caused by absorbing matter, the screen cannot be at a great distance, say, not more than 200 or 300 parsecs at most.

Looking next at the proper motions, it is seen that these are certainly not larger than those in surrounding regions. If, however, obscuring matter diminishes the brightness of the more distant stars to an appreciable extent, it would be expected that the brightest stars shown would be comparatively near, and that a larger proportion of the stars would have proper motions of considerable magnitude.



NUMBER OF STARS PER 100 SQUARE MINUTES TO MAGNITUDE 15·8 (PHOTOGRAPHIC).

The *dark areas* are shaded. Contour lines are drawn where the star density exceeds 20 and 40.

Alternatively the screen of absorbing matter may be assumed to be still nearer, so that practically all the stars in the region lie beyond it. In some respects this is the most reasonable, for the mass of gas which would be required to produce the necessary absorption (if it can be assumed that the absorption coefficient is of the same order as those determined for gases in the laboratory) will be very great, unless it is fairly near and of comparatively small extent.

8. The area in question has been considered by Prof. Barnard (*Ap. J.*, **25**, 3, "On a Nebulous Groundwork in the Constellation Taurus"). He concludes that the vacant spaces are due to absorption, and the appearance of the dark streaks on his photographs, taken with an exposure of $5^h\ 28^m$, as well as the presence of bright nebulæ, very strongly support his opinion. The sharpness of the boundary between the dense and sparse regions on the Franklin-Adams plates also supports this conclusion. On the other hand, the counts of stars and the proper motions would be more easily explained by a definite falling off in the number of stars in this region of the sky.

Is Venus Cloud Covered? By J. Evershed.

With reference to Mr. Clayden's note on the "Blue-Violet Absorption of Venus" (*M.N.*, **79**, 507), I would like to make the following observations :—

My suggestion that the atmosphere of Venus may be devoid of clouds was based on the surprising fact that in photographing the spectrum of the planet I found that a much longer exposure time was needed than was necessary to impress a spectrum when the image of a brightly illuminated cumulus cloud was brought on to the slit of the spectrograph, and this applies also to the green region of the spectrum.

As Venus is distant from the Sun 0·72 of an astronomical unit, the intensity of sunlight falling on the planet is 1·92 times greater than that received by the Earth at mean distance; but the double passage through the upper part of the Venus atmosphere will perhaps deplete the green light by as much as 30 per cent., assuming the atmosphere above her clouds to be somewhat more opaque than the terrestrial atmosphere above Mount Wilson, say at the 6000 feet level.* The ratio of exposure time for a terrestrial cloud and a Venus cloud will therefore be 1·3 to about 1, the absorption in the Earth's atmosphere being the same for both.

Actually I have not been able as yet to determine this ratio experimentally, owing to the very small number of days available in this moist climate when the sky is clear enough. All I can say is that a considerably longer time was found necessary for the

* For coefficients of atmospheric transmission see Abbot, *Annals of the Astrophysical Observatory of the Smithsonian Institution*, **3**, 138.

Venus spectrum than for the cloud spectrum when the altitude of Venus was greater than the altitude of the Sun when photographing the cloud.

The large discrepancy is either due to an absence or partial absence of clouds in Venus or, perhaps more probably, to the fact that owing to bad definition it is difficult, when Venus is not near full phase, to keep the brightest part of her disk continuously on the spectrograph slit during an exposure of about ten minutes; and when near full phase the small size of the image and low altitude of the planet impose still greater difficulties.

According to H. N. Russell, the reflective power of Venus is comparable with that which might be expected from a cloudy surface.* But this is perhaps not altogether conclusive, since it is deduced from indirect evidence of a not very certain nature.

If the atmosphere of Venus were cloudless but the lower strata heavily dust-laden, the reflective power would be less than for a cloudy atmosphere, and under these conditions it is almost certain that no surface features of the planet would be visible to us through the haze.

With regard to the absorption in the violet part of the spectrum, it is difficult to believe that the value of the colour index ($0^m{\cdot}78$) given by Russell† can be correct, in view of his statement on p. 188 of the article referred to below, that the Sun has a colour index of $+0^m{\cdot}79$; for this would mean that no selective absorption of violet light takes place in the planet's atmosphere above the reflecting layer of clouds or dust haze, and this is contrary to the evidence of my spectrum photographs, which show a marked falling off in relative intensity in the violet compared with the cloud spectrum.

Kodaikánal:
1919 *September* 30.

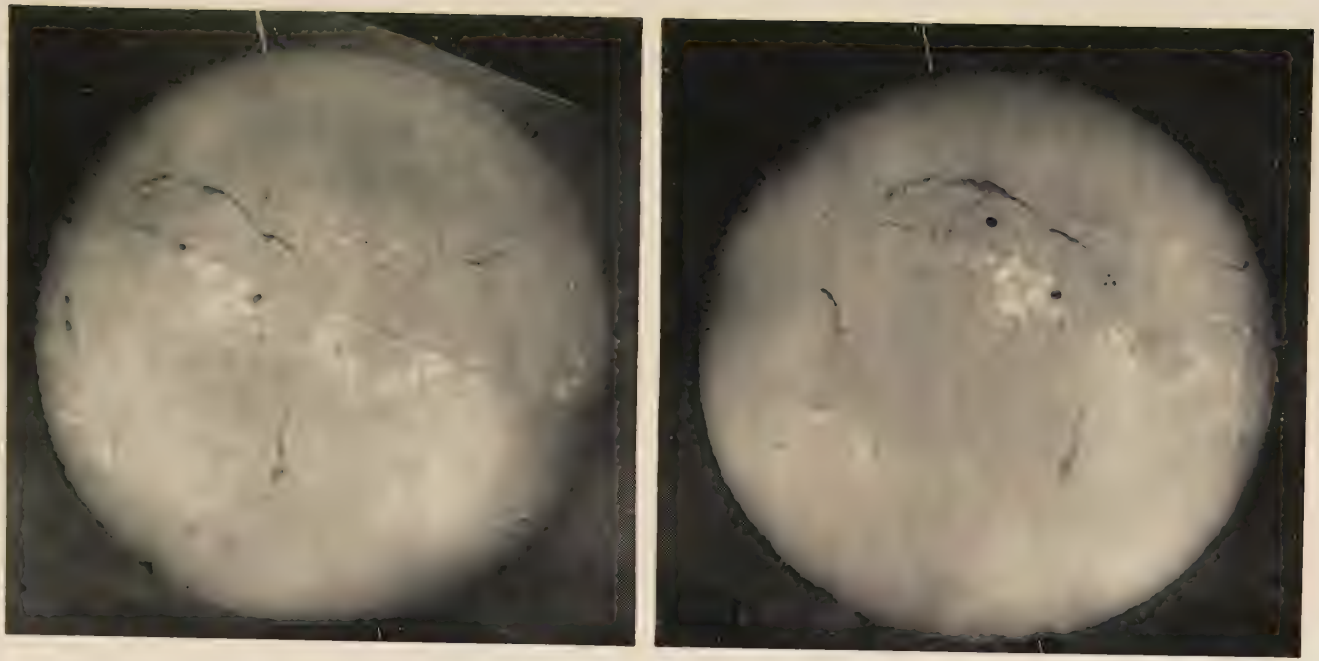
The Solar Prominence of 1919 *May* 29. By John Evershed. (Plates 2, 3.)

This prominence was photographed at Kodaikánal, and is referred to briefly in my report of prominence observations for the half-year ending 1919 June 30. As it was visible on the Sun's south-east limb during the total eclipse of May 29, a description of our photographs and of the past history of the prominence may be of use, especially for comparison with the interesting series of photographs secured with the Cambridge spectroheliograph and described by Mr. Moss (*M.N.*, 79, 553).

The prominence can scarcely be described as one of the eruptive

* *Astrophysical Journal*, 43, 193.
† *Ibid.*, 43, 190.

N



S

No. 1 1919 May 8, 8h 05m I.S.T.
or ,, 7, 14h 35m G.M.T

No. 2 1919 May 10, 8h 58m I.S.T.
or ,, 9, 15h 28m G.M.T.

Prominences seen by Absorption on the Sun's Disk in H_α Light.

class, although in the final stages it became partly separated from the Sun's limb and showed a tendency to rise, the greater part eventually dissolving away in space. But the early history of the prominence shows it to have been one of a stable character lasting through an entire rotation of the Sun. In the Kodaikánal series of photographs it is first seen on May 1, civil date, on the south-east limb. On the 2nd there is a series of prominences extending from $+25^\circ$ to -35° east, the maximum height attained being 60″ in the southern portion. On the 5th the prominences appear as absorption markings on the disc in Hα light both north and south of the equator. From this date Hα photographs show the prominences in patches or in continuous formation crossing the disc and maintaining their identity until May 14, when they are quite close to the west limb. The southern prominence is variable in length, extending from -15° to -30° on the 8th and 10th but inclined to a meridian about 15°, the southern end inclined towards the east (Plate 2, Nos. 1 and 2). It becomes much more extended both north and south on the 11th (Plate 3, No. 3), and appears to broaden out as it approaches near to the west limb; but this is an effect of perspective due to the considerable height of the prominence. The northern part crossed the central meridian on May 9 about $6\frac{1}{2}$ hours Indian standard time (May 8, 13^h G.M.T.). On the plate of May 11 it is seen to form the southern extension of an immense system of prominences reaching far into the northern hemisphere. The whole disposition of these prominences suggests an outer wave of a vast region of disturbance centring in a large sunspot the approximate co-ordinates of which are, north latitude 7° and longitude 7°. The southern prominence is nearly due south of this spot, and distant from it about 22° of latitude on May 10. In the north the prominences extend about 30° north of the spot, but trail away to the east through 33° of longitude. The southern prominence first appears projecting beyond the limb on May 15. On the 16th it is well developed between -15° and -43° and is 80″ in height (Plate 3, No. 4).

The next record of the prominence is eleven days later on the south-east limb, when portions of the prominence appear between -7° and -49°, bright but low. On May 28 it has enormously developed, extending almost continuously from -49° to $+22^\circ$ (Plate 3, No. 5); the southern portion is 140″ in height. On the 29th this part of the prominence is 280″ in height and is partly separated from the limb by a space 40″ to 60″ in height, but at -40° the prominence is attached to the chromosphere by a double stem or root (Plate 3, No. 6). The great mass of interlaced filaments are drawn out towards the north in two long streamers, the lower one connected to the limb near the equator and the higher ending in air, so to speak. It is interesting to compare this photograph with the Cambridge series, which show that this projecting streamer later extended itself towards the north, and finally made connection with the chromosphere 4° north of the

equator, the whole prominence on May 29, $2^h\ 29^m$ G.M.T., appearing as an immense arch 44° in extent. At Kodaikánal it was not possible to follow up the changes after $9^h\ 05^m$ on the 29th (May 28, $15^h\ 35^m$ G.M.T.) owing to clouds, but an opportunity occurred in the evening, when two disc plates were taken at $22^h\ 52^m$ and $23^h\ 03^m$ G.M.T. which show the prominence practically as it appears in the Cambridge series at $22^h\ 20^m$.

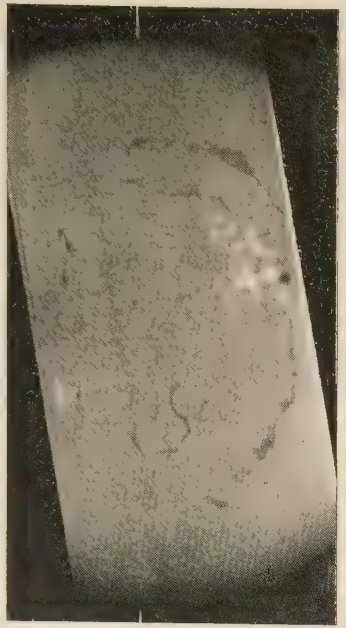
On May 30, $7^h\ 44^m$ I.S.T. (May 29, $14^h\ 14^m$ G.M.T.), the root of the prominence at −40° still remains, but the entire mass of filaments between that position and the equator has vanished.

In the plates the images are shown as seen in northern skies, with the north pole at the top and the east on the left side. The Sun's axis is indicated in the disc photographs, and the images of the limb are bisected along the axis.

It is interesting to note that the patches of absorption shown in No. 3 to the south of the spot correspond closely in position with the denser regions in the prominence at the limb as seen in No. 4 five days later.

The disc photographs in Hα light afford a good opportunity for measuring the angular rotation speed of the prominence, but an assumption has to be made as to the height of the absorption markings above the photospheric level. If this is taken at 30″, the daily angular motion near the limbs and near the centre of the disc give fairly consistent values, whereas if it is assumed to be at the photospheric level the daily motion near the limbs comes out greater than near the centre. Computing the angular motion therefore for a sphere of the Sun's radius + 30″, the mean daily motion between May 7 and May 13 is 13°·32. This is for the preceding end of the prominence in latitude −15°. Applying the appropriate correction for the Earth's movement, the sidereal angular speed is 14°·28 per diem, a value almost identical with the rotation movement of the reversing layer as determined by Adams for this latitude. It is to be noted also that the spot in north latitude 7° gives precisely the same angular motion, since it completed an entire synodical rotation in 27·02 days, according to measures of our photographs. The spot and the prominence therefore maintain throughout the same relative positions in longitude, and the prominence does not move with the higher speed which Adams found for the chromosphere.

No Hα disc photographs were obtained early in June owing to monsoon conditions, but on July 3, when the same region was near the central meridian, absorption markings are again found having much the same aspect as those recorded in May. There is a great mass in latitude −40°, which crosses the meridian on the 3rd and 4th. This may be identified with the root of the prominence of May 29, if we suppose a rotation at this latitude takes about a day longer than in latitude 15°. In addition to this there is a long filament between latitude −10° and −20° covering almost the same heliographic position as the prominence of May 29; that is, on July 3 it occupies the same position on the disc



No. 3. May 11, $11^h\ 03^m$ I.S.T.
or ,, 10, $17^h\ 33^m$ G.M.T.

On the disk in H_α.

No. 4. May 16, $8^h\ 26^m$ I.S.T.
or ,, 15, $14^h\ 56^m$ G.M.T.

On the west limb.

No. 5. May 28, $8^h\ 45^m$ I.S.T.
or ,, 27, $15^h\ 15^m$ G.M.T.

No. 6. May 29, $9^h\ 05^m$ I.S.T.
or ,, 28, $15^h\ 35^m$ G.M.T.

On the east limb.

The Solar Prominences of 1919 May 29.

above and in the work later carried out in our observatory, the units and the masses have been chosen in such a manner that $k^2m = 2$, $k^2m_1 = k^2m_2 = 1$. The differential equations for the relative motion then have the following form, when $\bar{x}_1$, $\bar{y}_1$, $\bar{x}_2$, $\bar{y}_2$ are the co-ordinates of m_1 and m_2 relative to m:

$$(2) \quad \cdot \quad \cdot \quad \cdot \quad \begin{cases} \dfrac{d^2\bar{x}_1}{dt^2} = -\dfrac{3\bar{x}_1}{\bar{r}_1^3} + \dfrac{\bar{x}_2 - \bar{x}_1}{\rho^3} - \dfrac{\bar{x}_2}{\bar{r}_2^3} \\ \dfrac{d^2\bar{y}_1}{dt^2} = -\dfrac{3\bar{y}_1}{\bar{r}_1^3} + \dfrac{\bar{y}_2 - \bar{y}_1}{\rho^3} - \dfrac{\bar{y}_2}{\bar{r}_2^3} \\ \dfrac{d^2\bar{x}_2}{dt_2} = -\dfrac{3\bar{x}_2}{\bar{r}_2^3} - \dfrac{\bar{x}_2 - \bar{x}_1}{\rho^3} - \dfrac{\bar{x}_1}{\bar{r}_1^3} \\ \dfrac{d^2\bar{y}_2}{dt^2} = -\dfrac{3\bar{y}_2}{\bar{r}_2^3} - \dfrac{\bar{y}_2 - \bar{y}_1}{\rho^3} - \dfrac{\bar{y}_1}{\bar{r}_1^3} \\ \bar{r}_1^2 = \bar{x}_1^2 + \bar{y}_1^2 \\ \bar{r}_2^2 = \bar{x}_2^2 + \bar{y}_2^2 \\ \rho^2 = (\bar{x}_2 - \bar{x}_1)^2 + (\bar{y}_2 - \bar{y}_1)^2. \end{cases}$$

The results given in the two above-mentioned papers have a certain interest as regards method, as it is probably the first time that any attempt has been made to integrate such a complicated system of differential equations with the aid of numerical integration. It is evident that, by the same method, we may, so long as there is no discontinuity in the problem, integrate any system of total differential equations with any number of variables and of any order and degree whatever.

Since the publication of the article in *Astr. Nach.*, 4356, much work has been done at our observatory with the aid of this method in searching for periodic solutions in the more general problem of three bodies.

At the same time we have continued the researches of Thiele, Burrau, and G. H. Darwin on the *restricted problem*. The results of these researches have been published in a series of papers by Burrau, Strömgren, Fischer-Petersen, and others. The scope of this work is a complete classification of the simple-periodic orbits in the restricted problem of three bodies with the proportion of masses that we have used ($m_1 = m_2$), and it is to be hoped that these researches will be finished at the end of this year.

In the present paper I propose to give a result that we have obtained during the first-mentioned work on the non-restricted problem, where the masses and the distances are both of the same order of magnitude.

§ 2. After several fruitless efforts, a special case of approximately periodic motion was found in the summer of 1919. In the computation and in the discussion preceding it, Mr. J. Fischer-Petersen, Miss J. M. Vinter-Hansen, and Messrs. P. Pedersen and N. Bennedsen have taken an essential part.

In the following table I give the co-ordinates of m_1 and m_2 (relative to m) which were found through numerical integration.

TABLE.

t.	$\bar{x}_1$.	$\bar{y}_1$.	$\bar{x}_2$.	$\bar{y}_2$.
− 2	−6·93868	+1·06853	+ 9·99710	−0·46126
− 1	−6·99317	+0·35720	+10·01013	−0·15360
0	−6·99317	−0·35720	+10·01013	+0·15360
1	−6·93868	−1·06853	+ 9·99710	+0·46126
2	−6·83060	−1·77078	+ 9·97072	+0·77032
3	−6·67084	−2·45820	+ 9·93043	+1·08157
4	−6·46201	−3·12550	+ 9·87542	+1·39569
5	−6·20740	−3·76799	+ 9·80465	+1·71305
6	−5·91084	−4·38162	+ 9·71694	+2·03391
7	−5·57646	−4·96306	+ 9·61103	+2·35817
8	−5·20862	−5·50967	+ 9·48561	+2·68552
9	−4·81175	−6·01954	+ 9·33936	+3·01536
10	−4·39021	−6·49118	+ 9·17099	+3·34685
11	−3·94825	−6·92395	+ 8·97931	+3·67893
12	−3·48995	−7·31750	+ 8·76319	+4·01032
13	−3·01914	−7·67191	+ 8·52161	+4·33955
14	−2·53937	−7·98764	+ 8·25371	+4·66497
15	−2·05398	−8·26542	+ 7·95873	+4·98478
16	−1·56603	−8·50618	+ 7·63606	+5·29700
17	−1·07828	−8·71111	+ 7·28529	+5·59953
18	−0·59328	−8·88149	+ 6·90608	+5·89014
19	−0·11333	−9·01872	+ 6·49839	+6·16645
20	+0·35952	−9·12432	+ 6·06239	+6·42598
21	+0·82343	−9·19987	+ 5·59811	+6·66610
22	+1·27679	−9·24702	+ 5·10640	+6·88407
23	+1·71818	−9·26746	+ 4·58797	+7·07707
24	+2·14643	−9·26292	+ 4·04396	+7·24218
25	+2·56052	−9·23515	+ 3·47580	+7·37637
26	+2·95972	−9·18589	+ 2·88534	+7·47664
27	+3·34339	−9·11690	+ 2·27482	+7·53996
28	+3·71127	−9·02990	+ 1·64696	+7·56335
29	+4·06320	−8·92653	+ 1·00493	+7·54400
30	+4·39931	−8·80838	+ 0·35247	+7·47930
31	+4·71990	−8·67688	− 0·30624	+7·36694
32	+5·02552	−8·53337	− 0·96646	+7·20514
33	+5·31690	−8·37896	− 1·62300	+6·99264
34	+5·59490	−8·21440	− 2·27037	+6·72885
35	+5·86052	−8·04048	− 2·90278	+6·41405
36	+6·11479	−7·85745	− 3·51437	+6·04935
37	+6·35874	−7·66530	− 4·09942	+5·63681
38	+6·59331	−7·46375	− 4·65245	+5·17942
39	+6·81932	−7·25225	− 5·16854	+4·68089
40	+7·03732	−7·03002	− 5·64430	+4·14598

The same co-ordinates are represented in fig. 1.

The diagram shows that the motion is approximately periodic. In the time from $t = -0{\cdot}5$ (the beginning of the integration) to $t = 37{\cdot}89$ the two masses m_1 and m_2 have moved in such a manner that the line m_1m_2, after having revolved about $138°{\cdot}7$, again passes through m, and so that the distances m_1m and mm_2 have approximately interchanged their values, now having their maximum and minimum respectively instead of the minimum and maximum which occurred at the beginning of the motion (at $t = -0{\cdot}5$). The exactly periodic orbit will be published in another connection.

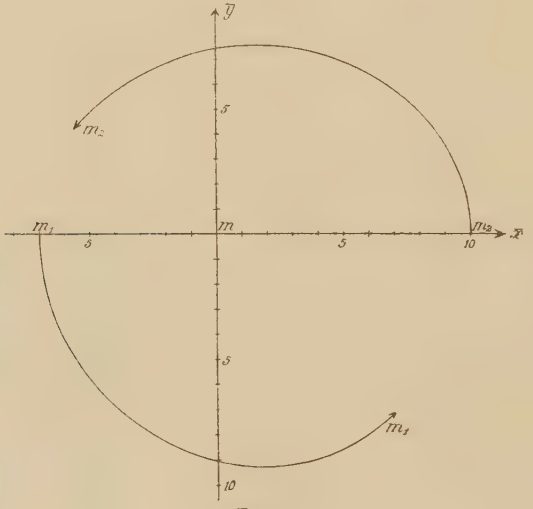


FIG. 1.

§ 3. In order to get a clearer idea of the nature of the motion, we make the following transformation :—From the initial position (at $t_0 = -0{\cdot}5$) up to the moment ($t_1 = +37{\cdot}89$), when all three masses are on the same straight line again, the time $t_1 - t_0 = 38{\cdot}39$ has elapsed. Instead of the old system of co-ordinates, with the origin in the mass m and with fixed directions of the axes, we now introduce a system with the origin in the common centre of gravity of the three masses and with axes rotating in the same sense as the line m_1m_2, with a speed of rotation which is identical with the mean speed of rotation of the line m_1m_2 in the time from $t = -0{\cdot}5$ to $t = 37{\cdot}89$.

With the aid of this transformation we get motions which are illustrated in fig. 2.

A mere glance at this design calls to mind the problems of libration in the restricted problem of three bodies. The special case of Lagrange, called the straight-line case, tells us that the

three masses m_1, m, m_2 can be placed in three points on a straight line (the proportions of the distances being functions of the masses) in such a way that, if we give them appropriate velocities, they will continue to lie on a straight line rotating around the common centre of gravity of the system in such a manner that the proportions of distances will be conserved. If we apply this case to the restricted problem of three bodies, we can express it

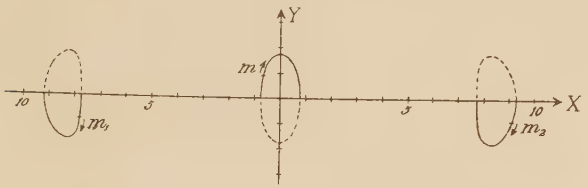


FIG. 2.

in the following way: if the two finite masses are given, it is possible to find for the infinitesimal body (m) such position on the rotating line through m_1 and m_2 that m will remain permanently in the same position. There exist three such points on the line through m_1 and m_2; these points are called points of libration (L_1, L_2, and L_3; L_3 to the left of m_1, L_2 to the right of m_2, and L_1 between m_1 and m_2).

Thiele and Burrau, following an idea of Gyldén, treated the problem of the motion of the infinitesimal mass *m in the neighbourhood* of the points of libration L_2 and L_3. They showed that there exist periodic motions in the neighbourhood of these points,

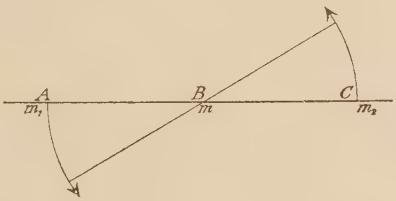


FIG. 3.

and the continuation of the first computed librations around L_2 and L_3 led Burrau to the important conception of *orbits of ejection* from the finite masses.* The corresponding researches for the point of libration L_1 are published in *Astr. Nachr.*, No. 4721.

§ 4. In order to decide whether the above-given approximately periodic solution (fig. 2) corresponds to the librations in the restricted problem of three bodies just mentioned, I have developed the theory of the infinitesimal periodic motions around the points, which in our more general problem ($m = 2m_1 = 2m_2$) correspond to the points of libration in the restricted problem.

* Cf. *Astr. Nachr.*, No. 3251.

As starting-point we take the circular motion which the two equal masses m_1 and m_2 perform around m (*i.e.* the common centre of gravity of the system) if we give m_1 and m_2 the appropriate velocities perpendicular to the line AC in fig. 3.

In the system with rotating axes the three masses will remain for ever at the three points A, B, C. Our problem is then to examine whether, for the three masses, there exist periodic solutions in *the neighbourhood* of A, B, and C.

In the purely circular motion we have for the co-ordinates of the three masses (Ξ, H, Ξ_1, H_1, Ξ_2, H_2) in a system with fixed directions and with the origin in m:

$$(3)\quad \left\{\begin{aligned} \Xi &= 0 \\ H &= 0 \\ \Xi_1 &= -\cos nt \\ H_1 &= -\sin nt \\ \Xi_2 &= \cos nt = -\Xi_1 \\ H_2 &= \sin nt = -H_1 \end{aligned}\right.$$

where $n = \frac{3}{2}$.

If for the actual motion we introduce the co-ordinates $\Xi + \xi$, $H + \eta$, etc., we get the following equations of motion:

$$(4)\quad \left\{\begin{aligned} \frac{d^2\xi}{dt^2} &= \frac{\Xi_2 + \xi_2 - \xi}{\rho_{02}^3} + \frac{\Xi_1 + \xi_1 - \xi}{\rho_{01}^3} \\ \frac{d^2\eta}{dt^2} &= \frac{H_2 + \eta_2 - \eta}{\rho_{02}^3} + \frac{H_1 + \eta_1 - \eta}{\rho_{01}^3} \\ \frac{d^2(\Xi_1 + \xi_1)}{dt^2} &= 2\frac{(\xi - \Xi_1 - \xi_1)}{\rho_{01}^3} + \frac{\Xi_2 + \xi_2 - \Xi_1 - \xi_1}{\rho_{12}^3} \\ \frac{d^2(H_1 + \eta_1)}{dt^2} &= 2\frac{(\eta - H_1 - \eta_1)}{\rho_{01}^3} + \frac{H_2 + \eta_2 - H_1 - \eta_1}{\rho_{12}^3} \\ \frac{d^2(\Xi_2 + \xi_2)}{dt^2} &= -2\frac{(\Xi_2 + \xi_2 - \xi)}{\rho_{02}^3} - \frac{\Xi_2 + \xi_2 - \Xi_1 - \xi_1}{\rho_{12}^3} \\ \frac{d^2(H_2 + \eta_2)}{dt^2} &= -2\frac{(H_2 + \eta_2 - \eta)}{\rho_{02}^3} - \frac{H_2 + \eta_2 - H_1 - \eta_1}{\rho_{12}^3} \end{aligned}\right.$$

We will now consider ξ, η, etc., as small quantities of the first order. For the distances ρ_{01}, ρ_{02}, ρ_{12} we then get the following expressions:

$$(5)\quad \left\{\begin{aligned} \rho_{01}^{-3} &= [(\Xi_1 + \xi_1 - \xi)^2 + (H_1 + \eta_1 - \eta)^2]^{-\frac{3}{2}} \\ &= 1 - 3(\Xi_1\xi_1 + H_1\eta_1) + 3(\Xi_1\xi + H_1\eta) \ldots \\ &= 1 + 3(\Xi_2\xi_1 + H_2\eta_1) - 3(\Xi_2\xi + H_2\eta) \ldots \\ \rho_{02}^{-3} &= 1 - 3(\Xi_2\xi_2 + H_2\eta_2) + 3(\Xi_2\xi + H_2\eta) \ldots \\ \rho_{12}^{-3} &= [(2\Xi_2 + \xi_2 - \xi_1)^2 + (2H_2 + \eta_2 - \eta_1)^2]^{-\frac{3}{2}} \\ &= \tfrac{1}{8}\{1 - \tfrac{3}{2}(\Xi_2\xi_2 + H_2\eta_2) + \tfrac{3}{2}(\Xi_2\xi_1 + H_2\eta_1) \ldots\} \end{aligned}\right.$$

With the aid of the simple differential equations for the purely circular motion, where $\xi = \eta$, etc., $= 0$, we obtain from the system (4) the differential equations for ξ, η, etc.:

$$(6)\quad \begin{cases} \frac{d^2\xi}{dt^2} = -3\Xi_2(\Xi_2\xi_2 + H_2\eta_2) - 3\Xi_2(\Xi_2\xi_1 + H_2\eta_1) + 6\Xi_2(\Xi_2\xi + H_2\eta) + \xi_2 + \xi_1 - 2\xi \\ \frac{d^2\eta}{dt^2} = -3H_2(\Xi_2\xi_2 + H_2\eta_2) - 3H_2(\Xi_2\xi_1 + H_2\eta_1) + 6H_2(\Xi_2\xi + H_2\eta) + \eta_2 + \eta_1 - 2\eta \\ \frac{d^2\xi_1}{dt^2} = \frac{5}{8}\Xi_2(\Xi_2\xi_1 + H_2\eta_1) - 6\Xi_2(\Xi_2\xi + H_2\eta) - \frac{3}{8}\Xi_2(\Xi_2\xi_2 + H_2\eta_2) + \frac{1}{8}\xi_2 - \frac{17}{8}\xi_1 + 2\xi \\ \frac{d^2\eta_1}{dt^2} = \frac{5}{8}H_2(\Xi_2\xi_1 + H_2\eta_1) - 6H_2(\Xi_2\xi + H_2\eta) - \frac{3}{8}H_2(\Xi_2\xi_2 + H_2\eta_2) + \frac{1}{8}\eta_2 - \frac{17}{8}\eta_1 + 2\eta \\ \frac{d^2\xi_2}{dt^2} = \frac{5}{8}\Xi_2(\Xi_2\xi_2 + H_2\eta_2) - 6\Xi_2(\Xi_2\xi + H_2\eta) - \frac{3}{8}\Xi_2(\Xi_2\xi_1 + H_2\eta_1) - \frac{17}{8}\xi_2 + \frac{1}{8}\xi_1 + 2\xi \\ \frac{d^2\eta_2}{dt^2} = \frac{5}{8}H_2(\Xi_2\xi_2 + H_2\eta_2) - 6H_2(\Xi_2\xi + H_2\eta) - \frac{3}{8}H_2(\Xi_2\xi_1 + H_2\eta_1) - \frac{17}{8}\eta_2 + \frac{1}{8}\eta_1 + 2\eta. \end{cases}$$

With the aid of

$$(7)\quad \begin{cases} \xi = X\cos nt - Y\sin nt \\ \eta = X\sin nt + Y\cos nt, \text{ etc.,} \end{cases}$$

we transform these equations to rotating axes.

The result is:

$$(8)\quad \begin{cases} \frac{d^2X}{dt^2} - 2n\frac{dY}{dt} - n^2X = 4X - 2X_1 - 2X_2 \\ \frac{d^2Y}{dt^2} + 2n\frac{dX}{dt} - n^2Y = -2Y + Y_1 + Y_2 \end{cases}$$

$$(8)_1\quad \begin{cases} \frac{d^2X_1}{dt^2} - 2n\frac{dY_1}{dt} - n^2X_1 = -4X + \frac{17}{4}X_1 - \frac{1}{4}X_2 \\ \frac{d^2Y_1}{dt^2} + 2n\frac{dX_1}{dt} - n^2Y_1 = 2Y - \frac{17}{8}Y_1 + \frac{1}{8}Y_2 \end{cases}$$

$$(8)_2\quad \begin{cases} \frac{d^2X_2}{dt^2} - 2n\frac{dY_2}{dt} - n^2X_2 = -4X - \frac{1}{4}X_1 + \frac{17}{4}X_2 \\ \frac{d^2Y_2}{dt^2} + 2n\frac{dX_2}{dt} - n^2Y_2 = 2Y + \frac{1}{8}Y_1 - \frac{17}{8}Y_2. \end{cases}$$

The problem of periodic solutions can now be solved in the following way. The integrals of the centre of gravity give the equations:

$$(9)\quad \begin{cases} 2X + X_1 + X_2 = 0 \\ 2Y + Y_1 + Y_2 = 0. \end{cases}$$

By (8) and (9) we get:

$$(10)\quad \begin{cases} \dfrac{d^2X}{dt^2} - 2n\dfrac{dY}{dt} - n^2X = 8X \\ \dfrac{d^2Y}{dt^2} + 2n\dfrac{dX}{dt} - n^2Y = -4Y \end{cases}$$

where $n = \frac{3}{2}$.

We write:

$$(11)\quad \begin{cases} X = Ae^{\lambda t} \\ Y = Be^{\lambda t}. \end{cases}$$

The systems (10) and (11) give the following equations for the determination of λ and of the quotient $\frac{B}{A}$:

$$A\lambda^2 - 3B\lambda - \tfrac{41}{4}A = 0$$
$$B\lambda^2 + 3A\lambda + \tfrac{7}{4}B = 0.$$

For the determination of λ we get:

$$(12)\quad (\lambda^2 - \tfrac{41}{4})(\lambda^2 + \tfrac{7}{4}) + 9\lambda^2 = 0,$$

and for $\frac{B}{A}$:

$$(13)\quad \frac{B}{A} = a = \frac{7 + 2\sqrt{2}}{\sqrt{1 + \sqrt{288}}}$$

The equation (12) has the following pure imaginary roots:

$$(14)\quad \lambda = \pm i\beta$$

where

$$\beta = \frac{\sqrt{1 + \sqrt{288}}}{2}$$

To (11), (13), and (14) the following periodic solution corresponds:

$$(15)\quad \begin{cases} X = K\cos\beta(t - t_0) \\ Y = -Ka\sin\beta(t - t_0). \end{cases}$$

Then we subtract $(8)_1$ from $(8)_2$ and get:

$$(16)\quad \begin{cases} \dfrac{d^2(X_2 - X_1)}{dt^2} - 2n\dfrac{d(Y_2 - Y_1)}{dt} - n^2(X_2 - X_1) = \frac{9}{2}(X_2 - X_1) \\ \dfrac{d^2(Y_2 - Y_1)}{dt^2} + 2n\dfrac{d(X_2 - X_1)}{dt} - n^2(Y_2 - Y_1) = -\frac{9}{4}(Y_2 - Y_1). \end{cases}$$

By the same method as before we deduce:

$$(17)\quad \begin{cases} X_2 - X_1 = C\cos\mu(t - t_1) \\ Y_2 - Y_1 = -2C\sin\mu(t - t_1) \end{cases}$$

where $\mu = n = \frac{3}{2}$.

If we now combine (9), (15), and (17) and write $M = \frac{1}{2}C$, we get the following solution of our problem:

$$(18) \quad \begin{cases} X = K \cos \beta(t - t_0) \\ Y = -K\alpha \sin \beta(t - t_0) \\ X_1 = -K \cos \beta(t - t_0) - M \cos n(t - t_1) \\ Y_1 = +K\alpha \sin \beta(t - t_0) + 2M \sin n(t - t_1) \\ X_2 = -K \cos \beta(t - t_0) + M \cos n(t - t_1) \\ Y_2 = +K\alpha \sin \beta(t - t_0) - 2M \sin n(t - t_1) \end{cases}$$

$$\alpha = 2{\cdot}318479 \ldots$$
$$\beta = 2{\cdot}119585 \ldots$$
$$n = \tfrac{3}{2}.$$

K and M are constants of integration.

§ 5. We have then obtained the following result. There exist in our problem two independent periodic solutions:—

I. The first one (the M-terms) only concerns the masses m_1 and m_2. The motions are retrograde, and take place in two similar ellipses. The two masses m_1 and m_2 lie at every moment at two diametrically opposite points in the ellipses. *The period is the same* $\left(\frac{2\pi}{n} = \frac{4}{3}\pi\right)$ *as the period for the rotation of the system of co-ordinates.* The proportion of the two axes in the ellipses is $B : A = 2 : 1$.

II. The second periodic solution (the K-terms) concerns all three masses. The motions are retrograde, and the orbits are similar ellipses as in case I. The two masses m_1 and m_2 always lie at corresponding points of their orbits; the mass m is situated at the diametrically opposite point of the orbit. The period of the motion is $\frac{2\pi}{\beta}$, in units of the period of the rotation of the system of co-ordinates: $\frac{n}{\beta} = \frac{3}{2\beta}$.

Of these two periodic solutions the first one (solution I.) corresponds, as can be easily shown, *to elliptic motions of* m_1 *and* m_2 *around* m instead of circular orbits. It is therefore simply an illustration of the more general Lagrangean case: the finite masses moving in ellipses, not in pure circles, around the common centre of gravity, with varying distances (but constant proportion of distances).

The second solution (II.), on the other hand, *represents a new periodic solution of the problem of three bodies.* This solution in the general problem (all masses of the same order of magnitude) corresponds to the librations of Gyldén, Thiele, Burrau in the restricted problem.

In order to show that the approximately periodic solution

(fig. 2) that we have obtained by aid of numerical integration is the continuation of the infinitesimal librations for which the theory has just been given, we need only compare the following two quantities in the finite and in the infinitesimal libration: (1) the proportion of the period of the libration to the period of rotation of the system of co-ordinates; and (2) the quotient of the two axes in the infinitesimal and in the finite librations. This comparison gives the following results:—In the theory of the infinitesimal librations the system of co-ordinates rotates 127°·4, while the libratory motion accomplishes half a revolution. In the computed finite example we get the value 138°·7. The difference is about 9 per cent. The proportion of the axes in the infinitesimal ellipses is $\alpha = 2{\cdot}318$; in the computed finite motion we get (for the mass m) the value 2·334. The difference is smaller than 1 per cent.

I must reserve for later publication the examination of:—

1. The development of our librations at greater distances from the points of libration;
2. The extension of the theory of the equilateral Lagrangean solutions corresponding to the extension of the theory of the straight-line solutions given in this paper; and
3. The general problem, where the three masses have general instead of numerical values.

Observatory, Copenhagen:
1919 *September.*

Postscript.

After this paper was written Mr. P. Pedersen has solved the above-mentioned problem 3 for the equilateral Lagrangean case (and thereby also problem 2).

1919 *December* 2.

Note on the Physical Aspect of the Einstein Prediction.

By H. F. Newall, F.R.S.

The remarkable announcement made at the Royal Society's joint meeting with the Royal Astronomical Society by the Astronomer Royal and Professor Eddington on 1919 November 6 has already given an opportunity for offering congratulations on the successful accomplishment of an excellently well-planned piece of eclipse work of a novel kind. The clean-cut problem to be solved was dealt with in a splendidly energetic manner, and all homage is due to the observers for the promptness with which the results have been extracted and made public.

As to the results themselves, it behoves us to examine them in no hurried way; and it is certainly right that those who are

interested in attempts to reconcile the discoveries of modern physics in regard to gaseous matter with the observed phenomena of the surroundings of the Sun far beyond the limits ordinarily assigned to a simple gravitational atmosphere should call for a pause before the interpretation given to the recent results is accepted as beyond doubt legitimate.

That the search for the existence of a measurable deflection of beams of starlight which pass through the extended corona should give a positive result is no surprise. Quite apart from Einstein's prediction, it is to be expected as a simple result of optical refraction. But the assignment of the whole of the observed deflection to satisfying the claims of that prediction seems to me equivalent to denying the possible existence of optical refraction through what I will call the framework of the corona.

For my own part, I should feel disposed to admit that the Newtonian suggestion may have been verified and that the balance of the observed deflection is attributable to optical refraction. But even in regard to such an admission I wish to hold an open mind, and for the following reasons, which I set forth prematurely and consequently inadequately. I recall the words of George Green, "I have no faith in speculations of these kinds unless they can be reduced to exact analysis." But here are the reasons, analysis can be given when the requisite data are forthcoming:—

A hot, radiating, spherical sun *must*, according to modern physics, be regarded as being surrounded by a multitude of atoms, ions, molecules, and electrons in a field of force which, by virtue of the superposition of effects which we assign to (*a*) gravitation, (*b*) the complex electrical conditions, and (*c*) radiation pressure, will be considerably different from a simple gravitational field. A purely gravitational atmosphere can extend only a barely perceptible distance from the sun as seen from the earth. But an atmosphere controlled by *gravity modified by radiation pressure and the electrical conditions* may extend to relatively large distances from the radiating surface of the sun. Indeed, on such an extended atmosphere we depend for the explanation of the phenomena of the prominences and the corona. The picture which I form to myself for thinking purposes is something like this: the extended solar atmosphere is probably nearly spherical or oblately spheroidal; and the structure of the visible corona—obviously essentially radial rather than spheroidal—is mainly due to local disturbances connected with spot centres, to arches, and to streams of matter and electrons and α-particles projected from disturbed localities on the active radiating solar surface and directed to some extent by the magnetic field of the Sun. The structure is picked out by enhanced luminosity or possibly localised absorption, or more probably both; but it is, in my mental picture, produced in an extended spheroidal framework which is not necessarily limited in its extent to the coronal structure which we can photograph.

Radiation pressure acting by virtue of the absorption exercised

by different molecules, atoms, and particles will produce effects by modifying gravitation in different degrees, according to the characteristics of the absorbing qualities of the affected matter. These molecules, etc., have their gravitating power most modified (within obvious limits) by radiation pressure, which have the most effective absorptive characteristics. Those, then, will have the best chance of surviving in the outer regions of the extended atmosphere, not only among molecules, etc., ejected from the Sun, but also among those which may be trying to come into the Sun. It is just such molecules which in bulk will have the best chance of contributing to a refractive index. The words "in bulk" are here inserted because we cannot say that what is lost in sparsity in volume distribution is made up in length of light path. The existence of a refractive index depends, according to modern physics, upon a volume density. The Helmholtz-Ketteler treatment of the question indicates what is to be expected; and Becquerel's and Wood's experiments on sodium vapour show that very appreciable refractive effects may be found even far from the absorption lines. We are hardly yet in a position to specify the departure of μ from unity in the case of vapours like iron, titanium, calcium, which have very numerous absorption frequencies in the visible and photographable spectrum; but there are no grounds for ruling out the possible existence of very appreciable refractive effects.

It must be admitted necessarily that until we can state quantitatively the laws of distribution and the calculable refractive effects of such matter, we have no right to claim an optical refraction. But, thanks to the recent observations, we have now for the first time got evidence of a deflection. It would be as unphilosophical to claim it wholly for refraction as to claim it wholly for the Einstein effect. It will require observations of an even higher order of precision than that of those recently made, to disentangle the deflection of Einstein from the deflection of refraction.

So far as to the Sun's action on *gaseous* matter. The gravitational action on *molar matter* is quite another story. In my mental picture, when a comet comes into the extended atmosphere, direct and nearly pure gravitation acts on the molar parts and sways them according to the law of Newton, modified, it may be, by reason of a resisting medium; the gaseous parts are, as is well known, swayed in various non-gravitational ways according to their various characteristic properties. I can see no reason to think that this differential action can be ruled out; the phenomena of comets, in respect both of the molar parts of the head and of the molecular parts of the tail, are proofs of its existence. Those who urge that comets which pass close to the Sun afford no evidence of an extended atmosphere or even afford evidence that there is no such atmosphere, seem to me to overlook the work of Backlund on Encke's Comet, as well as the investigations which tend to show that the orbit of a comet is not necessarily symmetri-

cal with respect to perihelion. The much-vaunted comet of 1843 was, as far as I recall the records, only observed after perihelion passage; and a reference to Galle's *Verzeichniss* shows that the periods assigned to it by various astronomers included the following:—$6\frac{1}{3}$, 175, 21·875, 36·9, 533, $147 = 4 \times 36{\cdot}9$ *years* in each case. What is the value of such evidence?

In connection with my references to refractive index and absorption of vapours, I may say that I had prepared a spectrograph especially to test a point of importance in this eclipse of May 1919. Professor Eddington was kind enough to take the instrument to Principe, but, as it turned out, he was not able to fit it into his observations; and even had he done so, I fear the clouds which he experienced would have spoiled the results. Still it will serve to show that I was keen to put my views to a crucial test. The object of the experiment was to see if it were possible to find evidence of selective absorption in the solar vapours at a considerable distance from the Sun's limb. My reasoning was as follows: κ_1 and κ_2 Tauri are recorded as stars of spectral type A, and thus their spectra have but faint indications of metallic absorption. Will they appear of solar type (metallic absorption dominant) when their spectra are photographed through the corona, and will their spectra revert to A type when the corona has passed by? If so, the solar characteristics must have been imposed by the absorption of the hypothetical outer solar atmosphere. If that atmosphere can absorb, it can also refract. I hope to have a chance of trying the experiment in Australia in September 1922. It is useless, I may point out, to try the experiment otherwise than during an eclipse, for the sunlight scattered by the earth's atmosphere would completely destroy its crucial nature.

The position, then, appears to me to stand thus:

1. Shall we accept the recent results as final verification of Einstein's prediction, so exact that we may cease to look for evidence of refraction in the corona?
2. Shall we accept the recent results as indicating the Newtonian half deflection, and search for evidence of optical refraction?

There seems to be a very natural disposition in many minds to accept "yes" as an answer to (1) because the motion of the perihelion of Mercury can be explained on the grounds underlying the prediction. But in leaning towards the other side, I venture to ask: Is it entirely out of the question that the very peculiarities in the properties and distribution of an extended atmosphere, which is swayed by gravitation modified selectively by radiation pressure, might be found capable of affording on Newtonian principles an explanation of the motion of the perihelion of Mercury without affecting overmuch the elements of Venus?

Cambridge:
1919 *November* 14.

On the Ellipticities of the Maclaurin Ellipsoids.
By H. C. Plummer.

1. The relation which connects the shape of a Maclaurin ellipsoid with its state of rotation is well known and may be written in the form

$$h = \frac{\cos\theta}{\sin^3\theta}\left[\theta\,(1+2\cos^2\theta) - 3\sin\theta\cos\theta\right] . \qquad (1)$$

where, in a notation easily understood,

$$h = \frac{\omega^2}{2\pi\gamma\rho}, \qquad \cos\theta = \frac{c}{a}, \qquad 0<\theta<\tfrac{1}{2}\pi.$$

The nature of the relation is not obvious at first sight from the equation. But (1) may be also expressed thus :—

$$h = \frac{\cos\theta}{\sin^3\theta}\left[(\theta-\sin\theta)(1+2\cos^2\theta)+\sin\theta\,(1-\cos\theta)(1-2\cos\theta)\right]$$

$$= \frac{\cos\theta}{1+\cos\theta}\left[\left(\frac{\theta}{\sin\theta}-1\right)\frac{1+2\cos^2\theta}{1-\cos\theta}+1-2\cos\theta\right]$$

Now there is a known series for $\theta/\sin\theta$, namely $(\theta<\pi)$,

$$\theta/\sin\theta = 1+\frac{1}{3}(1-\cos\theta)+\frac{1.2}{3.5}(1-\cos\theta)^2+ \ . \ . \ . \qquad (2)$$

Let

$$k = 1-\cos\theta = (a-c)/a, \qquad 0<k<1,$$

so that k is the ellipticity. Then

$$h = \frac{1-k}{2-k}\left[\left(\frac{1}{3}+\frac{1.2}{3.5}k+ \ . \ . \ .\right)\left(3-4k+2k^2\right)-1+2k\right]$$

$$= \frac{k(1-k)}{2-k}\left[\frac{16}{15}+\frac{32}{105}k+\sum_{n=3}\frac{1.2 \ . \ . \ . \ (n-1)}{3.5 \ . \ . \ . \ (2n-1)}\cdot\frac{3n^2+7n+6}{(2n+1)(2n+3)}k^{n-1}\right]$$

$$= \frac{8k(1-k)}{15(1-\frac{1}{2}k)}\left[1+\frac{1}{16}\sum_{n=2}\frac{1.2 \ . \ . \ . \ (n-1)}{7.9 \ . \ . \ . \ (2n+3)}(3n^2+7n+6)k^{n-1}\right]$$

$$= \frac{8k(1-k)}{15(1-\frac{1}{2}k)}\left(1+\frac{2}{7}k+\frac{3}{28}k^2+\frac{41}{924}k^3+ \ . \ . \ .\right) \ . \ . \ . \qquad (3)$$

$$< \frac{8k(1-k)}{15(1-\frac{1}{2}k)^2} = \frac{4}{15}\left\{1-\left(\frac{1-\frac{3}{2}k}{1-\frac{1}{2}k}\right)^2\right\} < 0{\cdot}27 .$$

The true limit of h is about 0·225.

2. The value of the series in (3) increases steadily throughout the range $0<k<1$ from 1 to about 1·5. For comparison with (1) shows that its value is

$$S = \frac{15}{16}\cdot\frac{1+\cos\theta}{1-\cos\theta}\cdot\frac{1}{\sin^3\theta}\left[\theta(1+2\cos^2\theta)-3\sin\theta\cos\theta\right]$$

$$= 15\,\pi/32,$$

when $\theta = \frac{1}{2}\pi$, $k = 1$. The denominator $(1 - \frac{1}{2}k)$ decreases from 1 to 0·5 in the same interval. Hence the factors $k(1-k)$ control the run of the function which is equal to h. It is evident that the function vanishes when $k = 0$ and when $k = 1$, and that it rises to a single maximum somewhere in the neighbourhoood of $k = \frac{2}{3}$. Above this maximum value of h there can be no spheroidal figure of equilibrium, and below it there are two distinct values of the ellipticity for a given value of h. When h is small, either k or $1 - k$ is small, and it is obvious that the lowest approximations are $k = 15h/8$, and $1 - k = 2h/\pi$, or $k = 1 - 2h/\pi$.

3. The inequalities

$$\frac{k(1-k)}{1-\frac{1}{2}k} < \frac{15}{8}h < \frac{k(1-k)}{(1-\frac{1}{2}k)^2}$$

may be used to facilitate the solution of (1) by trial, when h is not near its maximum value. Even without solving the quadratics, a second approximation gives

$$\tfrac{15}{8}h < k_1 < \tfrac{15}{8}h(1 + \tfrac{15}{16}h)$$

for the small root, and

$$1 - \tfrac{15}{16}h(1 + \tfrac{15}{8}h) < k_2 < 1 - \tfrac{15}{32}h(1 + \tfrac{45}{32}h)$$

for the large root. Thus if $h = 0{\cdot}1$, $0{\cdot}187 < k_1 < 0{\cdot}205$ and $0{\cdot}888 < k_2 < 0{\cdot}946$. The range for the large root is the greater, and the limits to search for solutions of (1) are $35^\circ{\cdot}6$ to $37^\circ{\cdot}3$ and $83^\circ{\cdot}6$ to $86^\circ{\cdot}9$.

4. The above inequalities suggest a form of approximation. For

$$(1 - \tfrac{1}{2}k)^{-4/7} = 1 + \tfrac{2}{7}k + \tfrac{11}{98}k^2 + \tfrac{33}{686}k^3 + \ \ldots$$

is a series which resembles the series S in (3) very closely. It is easily proved to be greater than S. Hence

$$h < \frac{8k(1-k)}{15(1-\frac{1}{2}k)^{11/7}},$$

but this represents a fairly close approximation. For the greatest possible value of h it gives

$$3k^2 - 24k + 14 = 0, \quad k = 0{\cdot}6337, \quad h < 0{\cdot}2254,$$

which is not greatly in excess of the true maximum.

5. This suggests the possibility of a closer approximation without greatly complicating the form. By giving the factor $(1 - k)$ an exponent different from 1, the first three terms of the series S can be exactly reproduced, and finding the maximum will still depend on the solution of a quadratic. In fact

$$(1-k)^{1/49}(1-\tfrac{1}{2}k)^{-30/49} = 1 + \tfrac{2}{7}k + \tfrac{3}{28}k^2 + \tfrac{57}{1372}k^3 + \ \ldots$$

$$= S - \tfrac{32}{1715}k^3 \ \ldots ;$$

Certainly for small values of k it can be inferred that

$$h < \tfrac{8}{15}k(1-k)^{50/49}(1-\tfrac{1}{2}k)^{-79/49}.$$

The greatest possible value of h is given by

$$k^2 - 8{\cdot}4k + 4{\cdot}9 = 0,\ k = 0{\cdot}6307,\ h > 0{\cdot}2242,$$

which comes very little below the true limit.

6. The actual maximum of (1) is found by simple differentiation and a little reduction to be given by the equation

$$\frac{2\theta}{\sin 2\theta} = \frac{8+\cos 2\theta}{5+4\cos 2\theta} \quad . \quad . \quad . \quad . \quad (4)$$

which can be solved by trial. The approximation corresponding to the above value of k is $\theta = 68^\circ{\cdot}3$. By 5-figure logarithms the nearest minute (51′) in 2θ is quickly found, and a final calculation with 8-figure logarithms gives

$$2\theta = 136^\circ\ 51'\ 20''{\cdot}94,\ \theta = 68^\circ\ 25'\ 40''{\cdot}47.$$

Hence

$$e = \sin\theta = 0{\cdot}92995567,\ k = 1-\cos\theta = 0{\cdot}63232838.$$

The function representing h is simplified by (4), and we have

$$\frac{\omega^2}{2\pi\gamma\rho} = h = \frac{\sin^2 2\theta}{5+4\cos 2\theta} = 0{\cdot}2246657 1.$$

The rate of approach of the logarithms of the two sides of (4) is rather low, and the last figure given is a little doubtful in every case.

7. An even more important point in the series of Maclaurin spheroids is the point of bifurcation to the Jacobi ellipsoids. This is given by the integration of an equation of Liouville on the assumption that the ellipsoid is one of revolution. The result may be expressed in the form

$$\frac{2\theta}{\sin 2\theta} = \frac{8-5\cos 2\theta}{4-\cos 4\theta} \quad . \quad . \quad . \quad . \quad (5)$$

where θ has the same meaning as before. The repeated root $\theta = 0$ is spurious, and results from the dropping of a factor $\cot^5\theta$. Using the series (2) with 2θ in the place of θ, and putting $1-\cos 2\theta = 2e^2$, we have

$$1 + \frac{2}{3}e^2 + \frac{2{\cdot}4}{3{\cdot}5}e^4 + \ldots = \frac{3+10e^2}{3+8e^2-8e^4}$$

or

$$1 + \frac{6}{7}e^2 + \frac{6{\cdot}8}{7{\cdot}9}e^4 + \ldots = \frac{5(1+2e^2)}{3+8e^2-8e^4}$$

when the first two terms of the series are transferred. Hence

$$\frac{1}{1-e^2} > \frac{5(1+2e^2)}{3+8e^2-8e^4} > \frac{1}{1-\frac{6}{7}e^2}$$

$$2e^4+3e^2-2>0>2e^4+8e^2-7$$

$$2e^2-1>0>2e^2+4-\sqrt{30}$$

$$1/\sqrt{2}<e<0{\cdot}86$$

$$45^\circ<\theta<59^\circ{\cdot}3.$$

This inequality limits the range within which a root of (5) can possibly lie.

8. The root would be expected nearer the upper than the lower limit, and the value of 2θ should be greater than 105°. Even 4-figure tables show that it is near $108^\circ{\cdot}8$, and 5-figure that it is near $108^\circ\ 43'$. The final calculation with 8-figure logarithms gives

$$2\theta = 108^\circ\ 42'\ 54''{\cdot}88,\ \theta = 54^\circ\ 21'\ 27''{\cdot}44.$$

The rate of approach of the logarithms of the two functions in (5) is very slow, and the form of solution may be compared with taking out 2θ from the logarithm of its cosine when its value is about 7°. No easy way of avoiding this difficulty suggests itself. The resulting values

$$e = \sin\theta = 0{\cdot}81266996,\ k = 1-\cos\theta = 0{\cdot}41727579,$$

are therefore distinctly uncertain in the last place, but the value of h, which is simplified by combining (5) with (1),

$$\frac{\omega^2}{2\pi\gamma\rho} = h = \frac{\sin^2 2\theta}{4-\cos 4\theta} = 0{\cdot}18711485,$$

should be within a unit in the last place. Darwin's 5-figure value for the eccentricity happens to be very close to the truth, but his value for h ($0{\cdot}18712$) is not so lucky.

9. In astronomical applications the constants are not the rate of rotation ω and the density ρ, but the mass and the moment of momentum,

$$M = \tfrac{4}{3}\pi\rho a^3\cos\theta,\ \mu = \tfrac{2}{5}Ma^2\omega.$$

Hence

$$h = h_1\cos\theta/a = h_2\rho^{\frac{1}{3}}\cos^{\frac{1}{3}}\theta,$$

where

$$h_1 = 25\mu^2/6\gamma M^3,\ h_2 = h_1(4\pi/3M)^{\frac{1}{3}}$$

are true constants for homogeneous masses undisturbed by tidal forces. The relation (3) now becomes

$$h_1/a = \frac{8k}{15(1-\frac{1}{2}k)}\left(1+\frac{2}{7}+\frac{3}{28}k^2+\ \ldots\right),$$

or

$$h_2\rho^{\frac{1}{3}} = \frac{8k}{15(1-\frac{1}{2}k)(1-k)^{\frac{1}{3}}}\left(1+\frac{2}{7}k+\frac{3}{28}k^2+\ \ldots\right),$$

of which the latter form (or rather some equivalent) is the more usual. In both cases it is obvious that the expressions on the right increase steadily as k increases from 0 to 1, and pass only once, for a single value of k, through any given value. The second expression tends to infinity, and no limit is imposed on the increase of the density. But the first expression tends to a finite limit $\frac{1}{2}\pi$ (§ 2), and therefore $\alpha > 2h_1/\pi$, or

$$\alpha > 25\mu^2/3\pi\gamma M^3,\ \omega < 9\pi^2\gamma^2 M^5/250\mu^3.$$

Of course these limits cannot be reached, or even remotely approached, but as a matter of curiosity it appears that the Earth would become a flat disc of radius 9·5 km., the Sun one of 6 km., while Jupiter and Saturn would have radii of 2680 and 4210 km. respectively. The corresponding rates of rotation are naturally very rapid—5 per second for the Earth, and over 6000 per second for the Sun. Jupiter would rotate in 50 seconds, and Saturn in 3 minutes as a limit.

10. The equation (1) can also be written in the form

$$\begin{aligned} h &= \cot^2\theta\left[\frac{2\theta}{\sin 2\theta}(1+2\cos^2\theta) - 3\right] \\ &= \cot^2\theta\left[\left(\frac{2\theta}{\sin 2\theta} - 1\right)\left(3 - 2\sin^2\theta\right) - 2\sin^2\theta\right] \\ &= \frac{1-e^2}{e^2}\left[\left(\frac{2}{3}e^2 + \frac{2\cdot 4}{3\cdot 5}e^4 + \ldots\right)\left(3 - 2e^2\right) - 2e^2\right] \end{aligned}$$

by using the same series as in § 7. Hence

$$\begin{aligned} h &= e^2(1-e^2)\sum_{n=1}\frac{2\cdot 4\ \ldots\ 2n}{3\cdot 5\ \ldots\ 2n+1}\cdot\frac{2n}{2n+3}e^{2n-2} \\ &= e^2\left(1-e^2\right)\left(\frac{4}{15} + \frac{32}{105}e^2 + \frac{32}{105}e^4 + \ldots\right). \end{aligned}$$

This is the counterpart of (3), but it does not convey so clearly the existence of a maximum value of h, or the double solution for a given value of h below this limit, because the series increases without limit as e approaches 1. On the other hand, when e is not large, the equation

$$\tfrac{15}{4}h = e^2 + \tfrac{1}{7}e^4 + 0.e^6 - \ldots$$

may be useful. Thus, to take a previous example, when $h = 0{\cdot}1$, the first two terms give $e^2 = 0{\cdot}3568$, $e = 0{\cdot}5973$, $\theta = 36°{\cdot}68$, $k = 0{\cdot}1980$, and these values are very little less than the true ones. This random example nearly represents the case of Saturn ($h = 0{\cdot}0983$), or rather of a homogeneous fluid body with the same angular velocity and mean density as Saturn. The departure of the calculated ellipticity (0·1944) from the observed (0·109) is of

course due to the condensation of matter towards the centre of this planet, which gives the largest value of h to be found in the solar system. In general

$$e^2 = \tfrac{15}{4}h/(1 + \tfrac{15}{28}h)$$

will give a sufficiently accurate approximation.

11. M. Hamy has proved that a fluid mass in which the density increases from the surface to the centre cannot rotate in equilibrium when all its homogeneous strata are bounded by coaxal ellipsoids. The following proof is on the same lines as that reproduced by Tisserand,* but is shorter and perhaps simpler in detail. Let the bounding surfaces be numbered 1, 2, . . ., n from the outer surface inwards, and let the densities of the corresponding strata be $\rho_1, \rho_2, \ldots, \rho_n$. The whole mass may be considered as made up of a complete ellipsoid of density ρ_1, another of density $\rho_2 - \rho_1$, and so on. Let the semi-axes of the ellipsoid i be a_i, b_i, c_i, its partial density $\rho_i - \rho_{i-1}$ ($\rho_0 = 0$). It will be convenient to write (ξ^1, η^1, ζ^1) as the co-ordinates of a point on the outer surface. Then the potential at this point is

$$V = \sum_{i=1}^{n} V_i$$

where

$$V_i = -C_i \int_{\lambda_i}^{\infty} \left(\frac{\xi}{a_i^2 + \lambda} + \frac{\eta}{b_i^2 + \lambda} + \frac{\zeta}{c_i^2 + \lambda} - 1 \right) \frac{d\lambda}{\Delta_i}$$

$$C_i = \pi\gamma(\rho_i - \rho_{i-1})a_i b_i c_i > 0, \quad \Delta_i^2 = (a_i^2 + \lambda)(b_i^2 + \lambda)(c_i^2 + \lambda)$$

$$\frac{\xi}{a_1^2} + \frac{\eta}{b_1^2} + \frac{\zeta}{c_1^2} = \frac{\xi}{a_i^2 + \lambda_i} + \frac{\eta}{b_i^2 + \lambda_i} + \frac{\zeta}{c_i^2 + \lambda_i} = 1 \quad . \quad . \quad . \quad (6)$$

Thus ζ is a function of ξ and η, and λ_i, defined as the positive root of the last equation, is also a function of ξ and η. Hence

$$0 = \frac{1}{a_1^2} + \frac{1}{c_1^2}\frac{\partial \zeta}{\partial \xi}$$

$$\left\{ \frac{\xi}{(a_i^2 + \lambda_i)^2} + \frac{\eta}{(b_i^2 + \lambda_i)^2} + \frac{\zeta}{(c_i^2 + \lambda_i)^2} \right\} \frac{\partial \lambda_i}{\partial \xi} = \frac{1}{a_i^2 + \lambda_i} + \frac{1}{c_i^2 + \lambda_i}\frac{\partial \zeta}{\partial \xi}$$

$$= \frac{1}{a_1^2}\left(\frac{a_1^2}{a_i^2 + \lambda_i} - \frac{c_1^2}{c_i^2 + \lambda_i} \right) \quad . \quad . \quad . \quad (7)$$

A necessary condition for equilibrium is that $V + \frac{1}{2}\omega^2(\xi + \eta) = \text{const.}$ over the ellipsoid (a_1, b_1, c_1). Hence

$$\frac{\partial^2 V}{\partial \xi^2} = \frac{\partial^2 V}{\partial \eta^2} = 0 .$$

* *Mécanique Céleste*, 2, 186-192.

By the elimination of ζ, V_i becomes

$$V_i = -C_i\int_{\lambda_i}^{\infty}\left[\frac{\xi}{a_1^2}\left(\frac{a_1^2}{a_i^2+\lambda}-\frac{c_1^2}{c_i^2+\lambda}\right)+\frac{\eta}{b_1^2}\left(\frac{b_1^2}{b_i^2+\lambda}-\frac{c_1^2}{c_i^2+\lambda}\right)+\frac{c_1^2}{c_i^2+\lambda}-1\right]\frac{d\lambda}{\Delta_i}$$

and therefore, since the integrand vanishes at the lower limit,

$$\frac{\partial^2 V_i}{\partial\xi^2} = -C_i\frac{\partial}{\partial\xi}\int_{\lambda_i}^{\infty}\frac{1}{a_1^2}\left(\frac{a_1^2}{a_i^2+\lambda}-\frac{c_1^2}{c_i^2+\lambda}\right)\frac{d\lambda}{\Delta_i}$$

$$= C_i\left\{\frac{\xi}{(a_i^2+\lambda_i)^2}+\frac{\eta}{(b_i^2+\lambda_i)^2}+\frac{\zeta}{(c_i^2+\lambda_i)^2}\right\}^{-1}\left(\frac{a_1^2}{a_i^2+\lambda_i}-\frac{c_1^2}{c_i^2+\lambda_i}\right)^2\frac{1}{a_1^4\Delta'_i}$$

by (7), where

$$\Delta'^2_i = (a_i^2+\lambda_i)(b_i^2+\lambda_i)(c_i^2+\lambda_i).$$

All the factors of this expression are essentially positive and a similar expression holds for $\partial^2 V_i/\partial\eta^2$. Hence $\partial^2 V/\partial\xi^2$ and $\partial^2 V/\partial\eta^2$ can only vanish, unless the whole mass is homogeneous, if

$$\frac{a_1^2}{a_i^2+\lambda_i}=\frac{b_1^2}{b_i^2+\lambda_i}=\frac{c_1^2}{c_i^2+\lambda_i}=1,\ i=2,\ 3.\ .\ .\ .,\ n,$$

the unit value being required by (6). Therefore the outer surface cannot be in equilibrium unless all the bounding surfaces of the strata are confocal.

12. When this condition is fulfilled, V_i becomes, on writing $\mu+\lambda_i$ for λ,

$$V_i = -C_i\int_0^{\infty}\left[\frac{\xi}{a_1^2}\left(\frac{a_1^2}{a_1^2+\mu}-\frac{c_1^2}{c_1^2+\mu}\right)+\frac{\eta}{b_1^2}\left(\frac{b_1^2}{b_1^2+\mu}-\frac{c_1^2}{c_1^2+\mu}\right)-\frac{\mu}{c_1^2+\mu}\right]\frac{d\mu}{\Delta_1}$$

where

$$\Delta_1^2 = (a_1^2+\mu)(b_1^2+\mu)(c_1^2+\mu).$$

Hence

$$\frac{\partial V_i}{\partial\xi} = -(a_1^2-c_1^2)C_i\int\frac{\mu d\mu}{a_1^2(a_1^2+\mu)(c_1^2+\mu)\Delta_1} \quad . \quad (8)$$

Let $(\xi'_{\frac{1}{2}}, \eta'_{\frac{1}{2}}, \zeta'_{\frac{1}{2}})$ be a point on the inmost surface n, so that

$$\frac{\xi'}{a_n^2}+\frac{\eta'}{b_n^2}+\frac{\zeta'}{c_n^2}=1 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The potential at this point is $V' = \Sigma V'_i$, where, when ζ' is eliminated,

$$V'_i = -C_i\int^{\infty}\left[\frac{\xi'}{a_n^2}\left(\frac{a_n^2}{a_i^2+\mu}-\frac{c_n^2}{c_i^2+\mu}\right)+\frac{\eta'}{b_n^2}\left(\frac{b_n^2}{b_i^2+\mu}-\frac{c_n^2}{c_i^2+\mu}\right)+\frac{c_n^2}{c_i^2+\mu}-1\right]\frac{d\mu}{\Delta_i}$$

$$\Delta_i^2 = (a_i^2+\mu)(b_i^2+\mu)(c_i^2+\mu) < \Delta_1^2$$

$$\frac{\partial V'_i}{\partial\xi'} = -C_i\int_0^{\infty}\frac{a_n^2(c_i^2+\mu)-c_n^2(a_i^2+\mu)}{a_n^2(a_i^2+\mu)(c_i^2+\mu)}\cdot\frac{d\mu}{\Delta_i}.$$

Now

$$\begin{aligned} &a_n^2(c_i^2+\mu)-c_n^2(a_i^2+\mu)\\ &=\mu(a_1^2-c_1^2)+(a_1^2-\lambda_n)(c_1^2-\lambda_i)-(c_1^2-\lambda_n)(a_1^2-\lambda_i)\\ &=(a_1^2-c_1^2)(\mu-\lambda_i+\lambda_n)=(a_1^2-c_1^2)(\mu+a_i^2-a_n^2). \end{aligned}$$

But a condition of equilibrium is that $V'+\frac{1}{2}\omega^2(\xi'+\eta')=\text{const.}$ over the ellipsoid (9). Hence

$$\tfrac{1}{2}\omega^2=-\frac{\partial V}{\partial\xi}=-\frac{\partial V'}{\partial\xi'} \qquad . \qquad . \qquad . \qquad (10)$$

By (8) $-\partial V_i/\partial\xi$ and therefore $-\partial V/\partial\xi$ have the sign of $(a_1^2-c_1^2)$. Hence $a_1>c_1$ and similarly $b_1>c_1$. Therefore

$$\begin{aligned} -\frac{\partial V_i'}{\partial\xi'}&=(a_1^2-c_1^2)C_i\int_0^\infty\frac{(\mu+a_i^2-a_n^2)d\mu}{a_n^2(a_i^2+\mu)(c_i^2+\mu)\Delta}\\ &>-\frac{\partial V_i}{\partial\xi}, \end{aligned}$$

for comparison with (8) shows that corresponding elements of the integrand have a larger factor in the numerator and respectively smaller factors in the denominator. Hence

$$-\frac{\partial V'}{\partial\xi'}>-\frac{\partial V}{\partial\xi},$$

which is incompatible with (10), and the only exception for a heterogeneous mass is when $a_1=b_1=c_1$, and $\omega=0$. This proves the theorem that the rotation of a heterogeneous mass in equilibrium is incompatible with complete ellipsoidal stratification.

The Short-Period Variable Y Lacertæ, with Remarks on the Phase Relation. By C. Martin and H. C. Plummer.

1. The variable Y Lacertæ was discovered by Mme. L. Ceraski * in 1906 on plates taken at Moscow by S. Blajko. Its discovery number was 23·1907, and other designations are B.D. $+50^\circ\cdot3581$, A.G.C. (Harvard) 7486, and No. 220550 on the Harvard system. It was observed by S. Blajko † himself, who determined the character and preliminary elements of the variation. Thirteen photometric observations by H. v. Zeipel ‡ a year later show a range of over half a magnitude. A longer series of visual estimations by N. Ichinohe § extends over the years 1907 and 1908, and gives improved elements and a light curve.

2. The period given by Ichinohe is $4^d\cdot3254$. Reduced with this period, the earlier Dunsink observations of 1913 fall quite distinctly late when compared with the observations of five years

* *A.N.*, 4159. † *A.N.*, 4196. ‡ *A.N.*, 4247. § *A.N.*, 4342.

later. The displacement of the ascending branch is $0^d{\cdot}780$ corresponding to an interval of 411 periods, implying a correction of $-0^d{\cdot}00190$ to the supposed period. Hence the elements now adopted,

$$\text{J.D. } 2421\ 818{\cdot}437 + 4{\cdot}32350\ E,$$

are an independent determination based on the Dunsink photographic observations alone. Carried backward over 972 periods they give a maximum at J.D. 2417 615·975, which is $0^d{\cdot}135$ later than Blajko's epoch and $0^d{\cdot}235$ later than Ichinohe's. The light curve of the latter shows a double maximum with an interval of about half a day, and the effect of the correction (if so regarded) is to transfer the epoch from the preceding maximum to a point roughly half-way between the two maxima. This cannot be considered as evidence that the period has really diminished in the last ten years. The character of the maximum may have changed or may be irregular. But this is only suggested by a comparison of photographic observations with visual estimates. The present observations do not suggest a double or even a protracted maximum, and give a curve resembling that of V Lacertæ,* whereas Ichinohe's visual curve recalls the Dunsink photographic curve for X Lacertæ, determined at the same time and from the same plates as that of V Lacertæ.

3. Information relating to the comparison stars is given in Table I. The magnitudes m_1, m_2 are based on Schwarzschild's

TABLE I.

B.D.+50°.	Pl. 1977. m_1	Pl. 1982. m_2	Mean. m	Corr.	Final. m'	P.E.	Range.
3582	8·88	8·97	8·92	−·006	8·91	±·012	·16
3594	9·45	9·55	9·50	+·011	9·51	±·020	·14
3560	9·88	9·92	9·90	−·021	9·88	±·020	·13
3589	10·06	10·12	10·09	+·008	10·10	±·012	·06
3588	10·25	10·32	10·29	+·010	10·30	±·017	·10

system for the χ Persei cluster, through two plates bearing duplicate exposures on this region and the region of Y Lacertæ. The mean magnitudes m have been adopted in the reduction of the variable, the plate residuals giving small mean corrections which are incorporated in m'. From the same residuals, of which the extreme range is given in the last column, the probable error for a single plate is inferred and shown in the preceding column.

4. The results from 56 exposures on the variable are given in Table II. In the second and third columns are recorded the astronomical date and G.M.T., together with the Julian day and decimal fraction, corresponding to the middle of each exposure, the length of which is shown in the fifth column. The phase in days, given in the fourth column, is reckoned from maximum and based

* *M.N.*, 76, 242.

TABLE II.

Serial No.	Date G.M.T.			Julian Day.	Phase in Days.	Exposure.	Magn.	Obs. —Curve.	Images.
	1913 d	h	m	2420		min.			
1	Oct. 30	8	34	071·357	3·94	30	9·53	+ ·04	Fair.
2	Nov. 21	9	36	093·400	0·04	30	9·31	+ ·01	Fair.
3	Dec. 23	6	23	125·266	1·64	30	9·80	− ·03	Fair.
4	23	7	56	125·331	1·71	30	9·90	+ ·04	Fair.
5	29	6	15	131·260	3·31	30	10·05	·00	Fair.
6	30	6	12	132·258	4·31	30	9·29	− ·01	Good.
7	31	6	16	133·261	0·99	30	9·70	+ ·05	Fair.
8	31	8	50	133·368	1·10	30	9·66	·00	Fair.
	1917			2421					
9	Dec. 10	6	10	573·275	1·28	40	9·63	− ·05	Good.
10	10	7	24	573·309	1·31	44	9·68	− ·01	Good.
11	10	8	13	573·342	1·34	40	9·65	− ·05	Good.
	1918								
12	Aug. 10	10	28	816·437	2·32	30	9·99	+ ·01	Good.
13	10	11	9	816·464	2·35	30	10·02	·00	Good.
14	12	10	18	818·430	4·32	30	9·31	+ ·01	Fair.
15	12	10	59	818·458	0·02	30	9·30	·00	Good.
16	12	11	28	818·478	0·04	30	9·27	− ·03	Good.
17	12	11	59	818·499	0·06	30	9·26	− ·05	Good.
18	13	10	17	819·429	0·99	30	9·66	+ ·01	Fair.
19	13	10	48	819·451	1·01	30	9·67	+ ·02	Good.
20	13	11	28	819·478	1·04	30	9·66	·00	Fair.
21	13	11	58	819·499	1·06	30	9·65	− ·01	Fair.
22	14	10	39	820·445	2·01	30	9·96	− ·02	Fair.
23	14	11	10	820·465	2·03	30	9·99	+ ·01	Fair.
24	28	9	35	834·400	2·99	30	10·00	− ·08	Fair.
25	29	10	57	835·457	4·05	30	9·40	+ ·01	Fair.
26	31	10	34	837·441	1·71	30	9·89	+ ·03	Fair.
27	31	11	5	837·462	1·73	30	9·90	+ ·03	Good.
28	Sept. 2	9	58	839·416	3·69	30	9·78	+ ·01	Good.
29	2	10	28	839·437	3·71	30	9·76	+ ·03	Fair.
30	2	11	6	839·462	3·73	30	9·80	+ ·09	Good.
31	10	9	5	847·379	3·00	30	10·06	− ·03	Good.
32	11	9	3	848·377	4·00	27	9·46	+ ·02	Fair.
33	11	11	11	848·465	4·09	30	9·34	− ·03	Good.
34	12	10	23	849·433	0·73	38	9·56	− ·04	Good.
35	26	8	16	863·344	1·67	30	9·87	+ ·03	Fair.
36	26	10	3	863·419	1·75	30	9·88	+ ·01	Fair.
37	27	9	54	864·413	2·74	24	10·06	+ ·01	Fair.

TABLE II.—*continued.*

Serial No.	Date G.M.T.	Julian Day.	Phase in Days.	Exposure.	Magn.	Obs. —Curve.	Images.
	1918 d h m	2421		min.			
38	Sept. 28 9 51	865·411	3·74	30	9·67	− ·03	Fair.
39	30 10 1	867·418	1·42	30	9·68	− ·04	Fair.
40	Oct. 7 9 26	874·393	4·07	40	9·40	+ ·03	Fair.
41	10 10 15	877·427	2·78	38	10·08	+ ·03	Fair.
42	23 7 10	890·299	2·69	40	10·06	+ ·02	Fair.
43	Nov. 5 7 40	903·320	2·74	40	10·07	+ ·02	Good.
44	5 10 49	903·451	2·87	40	10·06	− ·01	Fair.
45	6 10 43	904·448	3·86	40	9·56	− ·02	Good.
46	6 11 30	904·479	3·90	40	9·57	+ ·03	Good.
47	11 7 32	909·314	0·08	40	9·30	− ·01	Good.
48	11 10 15	909·428	0·20	30	9·33	− ·02	Fair.
49	26 8 42	924·363	2·16	40	9·97	− ·04	Fair.
50	Dec. 6 7 51	934·328	3·48	40	9·99	+ ·03	Good.
51	6 8 34	934·357	3·51	40	9·92	− ·02	Good.
52	6 10 48	934·451	3·62	40	9·88	+ ·04	Good.
	1919						
53	Jan. 28 10 16	987·428	0·37	60	9·41	− ·02	Fair.
		2422					
54	Feb. 23 9 7	013·379	0·38	60	9·47	+ ·04	Good.
55	23 10 20	013·431	0·44	60	9·49	+ ·02	Good.
56	23 14 9	013·593	0·60	60	9·55	·00	Fair.

on the elements adopted in § 2. The curve from which the differences shown in the seventh column are derived represents the equation

$$m = 9{\cdot}76 + 0{\cdot}34 \sin(\theta + 240^\circ) + 0{\cdot}11 \sin(2\theta + 282^\circ) + 0{\cdot}07 \sin(3\theta + 304^\circ)$$

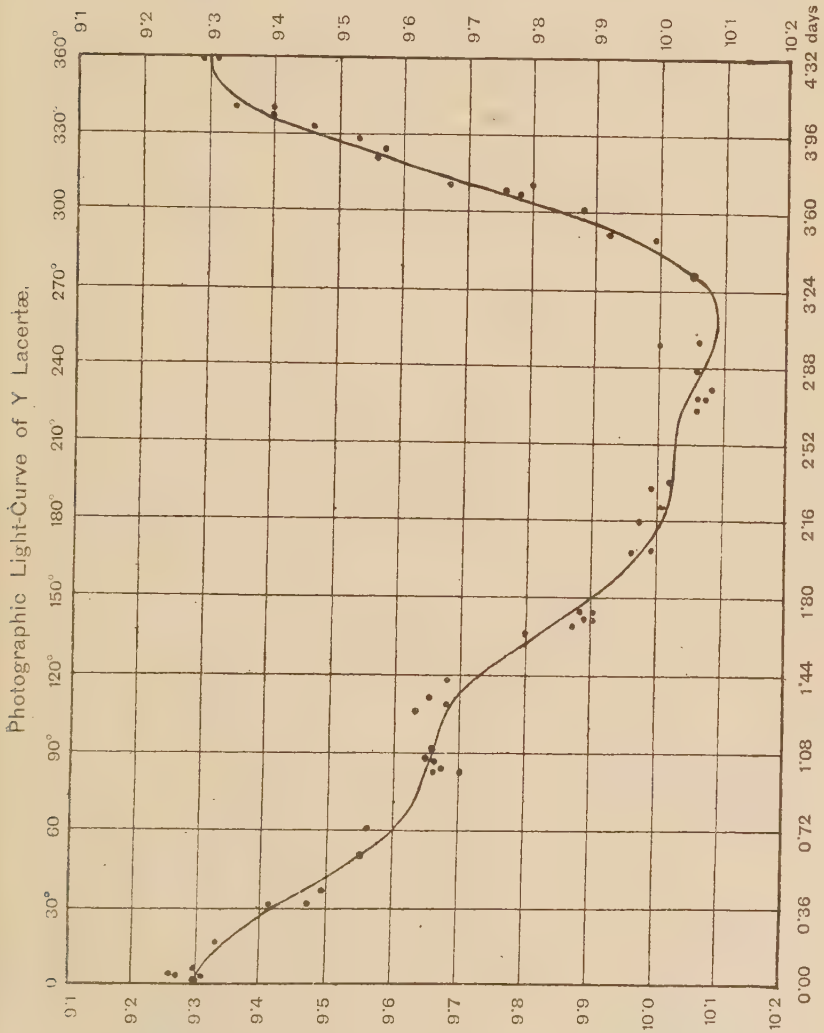
where θ has the value 2π corresponding to the period of the star and is reckoned from the assumed epoch of maximum. The residuals set down in this column indicate a probable error of $\pm 0{\cdot}020$ in a measured magnitude of the variable.

5. A comparison of the observations with the same curve is shown graphically in the accompanying figure. The equation of the curve is exceptionally simple, and the representation could not be materially improved by including terms beyond the third harmonic. By displacing the epoch it may be written in the form

$$m = 9{\cdot}76 - 0{\cdot}34 \cos\phi + 0{\cdot}11 \cos(2\phi - 108^\circ) + 0{\cdot}07 \cos(3\phi - 56^\circ).$$

Thus $\phi_2 = 108^\circ$, $\phi_3 = 56^\circ$, and $\phi_2 - 0{\cdot}6\phi_3 = 74^\circ$, so that the point (ϕ_2, ϕ_3) falls almost exactly on the phase line determined by Mr. Phillips for long-period stars of Group II.

6. It is now possible to extend slightly a table * previously given. This is done in Table III., which is based entirely on Dunsink material. The few additions hardly affect the earlier remark that ϕ_2 has a distinct tendency towards a constant value. It certainly



keeps within a very small range, and if all these stars, including the last two, are to be included in a single class, the facts taken by themselves suggest a new class characterised by $\phi_2 = \text{const.}$, rather than by the relation, now familiar, $\phi_2 - 0{\cdot}6\phi_3 = \text{const.}$ Either inter-

* *M.N.*, 78, 161.

TABLE III.

Star.	Period.	ϕ_1.	ϕ_2.	ϕ_3.	$\phi_2 - 0{\cdot}6\phi_3$.
	h	°	°	°	°
RZ Cephei . .	7·4	180	108	47	80
RS Boötis . .	9·1	,,	124	50	94
SW Andromedæ * .	10·6	,,	119	51	88
XZ Cygni . .	11·2	,,	133	75	88
RR Ceti . .	13·3	,,	104	39	81
RR Lyræ . .	13·6	,,	118	56	84
SW Draconis .	13·7	,,	130	26	114
SU Draconis . .	15·9	,,	118	58	83
	d				
Y Lacertæ . .	4·3	,,	108	56	74
V Lacertæ . .	5·0	,,	108	33	88
X Lacertæ . .	5·4	,,	111	121	(38)
RW Cassiopeiæ .	14·8	,,	116	105	(53)

pretation is, however, fairly open if the last two stars are excluded. The short-period variables analysed by Mr. Phillips † hardly help to decide the question. His short list requires us to admit a range of more than 50° in the value of $\phi_2 - 0{\cdot}6\phi_3$, and an equal range in ϕ_2 takes in all the stars except three, and their position in the list looks precarious. In particular, five out of his thirteen stars, including δ Cephei itself with $\phi_2 = 108°$, fall within the range $103° < \phi_2 < 124°$.

7. The regions of the clusters ω Centauri, M5 and M15 contain nine known variables with periods exceeding one day. As shown in Table IV., seven of these stars follow the now recognised rule,

TABLE IV.

Cluster.	Bailey's No.	Period.	ϕ_1.	ϕ_2.	ϕ_3.	$\phi_2 - 0{\cdot}6\phi_3$.
		d	°	°	°	°
ω Centauri	60	1·3	180	142	52	111
,,	1	29·3	,,	90	4	88
,,	2	484	,,	182	147	94
M5	42	25·7	,,	103	21	90
,,	84	26·5	,,	81	4	79
,,	50	106·0	,,	122	47	94
M15	1	1·4	,,	146	66	106

and that considerably better than the stars analysed by Mr. Phillips. The mean value of $\phi_2 - 0{\cdot}6\phi_3$ is 93°·6, and the range in ϕ_2 is considerable. The casual evidence of this table is rather striking in its suggestion that the phase rule is independent of period. Star 2 in ω Centauri is very abnormal, with a second harmonic

* The observations of this star will be published shortly.

† *M.N.*, 78, 186.

nearly three times as large as the first. Of the two stars in the same cluster now omitted, one (No. 42, period 297 d.) has a similar character, and the other (No. 29, period 14·75 d.) has no special peculiarity, but its third harmonic is small.

8. Among the isolated variable stars of short period it certainly seems difficult to get the same range in ϕ_2 as that covered by the variables in clusters. The distribution for the latter has been shown * diagrammatically for the three clusters ω Centauri, M3 and M5. In the cluster M15 the range for stars of sub-class *a*—113° to 163°—is much the same as in M5. As no diagram has been given for M15 it may be of interest to give the values of $\phi_2 - 0{\cdot}6\phi_3$ for the individual stars in this cluster. This is done in Table V., which is divided into two parts corresponding to the separate sub-classes, and arranged in groups of five reproducing

TABLE V. M15. $\phi_2 - 0{\cdot}6\phi_3$.

Sub-class *c*. Period $< 0{\cdot}436$ d.

92	90°	121°	83°	103°	121°
...	80	...	91	82	89
104	75	66	102	96	94
108	90	56	78	100	80
...	105	93	105	116	

26 stars. $\phi_2 - 0{\cdot}6\phi_3 = 92°{\cdot}8$ (mean).

Sub-class *a*. Period $> 0{\cdot}570$ d.

°	94°	97°	91°	83°	98°	95°
	105	92	91	99	108	89
	92	90	94	89	83	89
	86	100	84	84	82	88
101	106	92	90	91	95	

30 stars. $\phi_2 - 0{\cdot}6\phi_3 = 92°{\cdot}6$ (mean).

the arrangement of the original table. Only three stars of sub-class *c* fail to follow the phase relation approximately, and their third harmonics have the coefficients ·00 (<·005), ·01 and ·02 mag. In some cases the points (ϕ_2, ϕ_3) fall below (90°, 0°), and it is then necessary to take $\phi_3 - 360°$ instead of ϕ_3. The separation of the periods in the two groups is quite sharply marked in this cluster, but in spite of it the essential physical continuity of the stars is strongly suggested by the mean values of $\phi_2 - 0{\cdot}6\phi_3$. The greater scattering of the individual values in the first group may be only the result of the smaller amplitude in the light variation and the natural uncertainty in the determination of the small third harmonic.

9. A table similar to Table V. can be constructed for the stars of sub-class *c* in ω Centauri, for these also have been excluded from the diagrams. The values of $\phi_2 - 0{\cdot}6\phi_3$ are shown in Table VI. The degree of scattering is greater than in the correspond-

* *M.N.*, 79, 639.

ing group in Table V. A certain latitude in the interpretation of the proper quadrants is sometimes necessary, an extreme instance being $\phi_2 = 244^\circ - 360^\circ$, $\phi_3 = 48^\circ - 360^\circ$, $\phi_2 - 0{\cdot}6\phi_3 = 71^\circ$. Also the third harmonic is very small, its coefficient in no case exceeding

TABLE VI.

ω Centauri.

Sub-class *c*. Periods < 0·49 d.

°	°	°	°	°	°	
103	75	...	102	101	57	51
90	114	102	...	128	91	
87	88	106	72	78	71	
...	69	64	95	60	100	
73	50	117	103	...	...	

26 stars. $\phi_2 - 0{\cdot}6\phi_3 = 86^\circ{\cdot}4$ (mean).

0·04. Still, in spite of doubtful cases, the story is much the same, and the phase relation can again be traced when it is looked for. The periods have been arbitrarily restricted to the point at which the stars of sub-class *a* begin to appear. Otherwise stars of the group occur occasionally with all periods (contrary to the evidence of M 15), even among the stars assigned by Bailey to sub-class *b*.

10. Similarly the eight stars of shortest period in M5, which alone Bailey assigns to sub-class *c*, and for which he suggests doubling the period, do not really stand without parallels among the rest. But they are marked by two peculiarities. One is the pronounced gap (0·1 d.) in the periods between the last star of the group and the first star of the remainder. The other is the circumstance that not only is the third harmonic small but the second is exceptionally small also. Hence it was natural to consider both negligible, and this led to the idea that the stars might be attributed to the same type as RR Centauri. But though both ϕ_2 and ϕ_3 must be very uncertain, the eight values of $\phi_2 - 0{\cdot}6\phi_3$ are

$$53^\circ,\ 148^\circ,\ 118^\circ,\ 106^\circ,\ 65^\circ,\ 68^\circ,\ 21^\circ,\ 81^\circ,$$

with the mean 82°·5, and the deviations though large are not greater than ought reasonably to be expected. As a matter of fact, the two stars in the same cluster with the largest periods under one day probably belong to the same sub-class, and though it is asserted that no such stars have been found in M3, the statement can hardly be maintained if the magnitude scale is reduced by half. For example, we may take the two stars:

Star.	Period.	r_1.	r_2.	r_3.	ϕ_1.	ϕ_2.	ϕ_3.
M3 (100)	·558	·55	·15	·05	180°	144°	68°
ω Centauri (117)	·422	·26	·09	·03	180°	131°	66°

It is evident that the two curves must resemble one another closely except in scale, and the second is accepted as a member of sub-class *c*. The conclusion is that stars of the group in question favour the shortest periods and certain clusters, but that the preference is by no means exclusive. Indeed, if sub-class *c* be

distinguished by the relative smallness of the higher harmonics, especially the third, and the same property be looked for in the long-period variables, it will be seen that the majority of Group II. stars fall into this division, and that only a few, like S Tauri and R Aquarii, are comparable with the cluster stars of sub-classes *a* or *b* in the relative importance of their higher harmonics.

11. The relation found originally by Mr. Phillips for the long-period variables of Group II. was $\phi_3 = 1{\cdot}67\phi_2 - 126°{\cdot}8$ or $\phi_2 - 0{\cdot}6\phi_3 = 76°$. It will be noticed that in all the above partial determinations of the constant the mean value is distinctly greater than 76°, and generally not far from 90°. The value of the coefficient 0·6 has been accepted from the original investigation, and, indeed, it clearly accords with the evidence of the (ϕ_2, ϕ_3) diagrams for ω Centauri, M3 and M5 quite well. If the magnitude equation is written in the form

$$m = m_0 - r_1 \cos\phi + r_2 \cos 2(\phi - \phi'_2) + r_3 \cos 3(\phi - \phi'_3),$$

the phase relation becomes

$$2\phi'_2 - 1{\cdot}8\phi'_3 = \text{const.}$$

If, on the other hand, the coefficient could be changed from 0·6 to 2/3, it would take the simpler form

$$\phi'_2 - \phi'_3 = \text{const.},$$

which might be more suitable for a physical theory. Whether the change could be made consistently with all the evidence now existing is a question which can be left till the answer is required by an actual theory. In the meantime the coefficient 0·6 may be accepted for the large group of long-period variables for which it was originally proposed, and equally for the great majority of short-period variables.

The Long-period Variable W Cygni. By H. H. Turner and Mary A. Blagg.

1. The star W Cygni is given in the list in Table V. on p. 98 of *M.N.*, 78, as specially interesting because of its provisional $+\alpha$ and low galactic latitude.

On noticing this list, Mr. C. L. Brook wrote that a large number of observations of this star had been made by the Variable Star Section of the B.A.A., which he very kindly offered to send for examination. A preliminary discussion of the results of one observer (ANB) seemed to promise such close regularity, in spite of the comparatively small limits of variation, as to justify a careful and extended reduction, which was thereupon undertaken by Miss Blagg.

2. The preliminary work ultimately suggested for adoption a period of $129\frac{1}{3}$ days. But it soon became clear that there were distinct departures from regularity, notably the abrupt change of about 40 or 50 days in epoch between cycles 25 and 26. Hence in deducing a mean light curve, certain cycles were arbitrarily shifted. Since the mean light curve is chiefly an intermediary, the details need not be given. Below is given the mean light curve deduced from all six observers (weighted as in Table III.). The accordance is good.

TABLE I.

Mean Light Curve for W Cygni in Units of 0·01 *mag.*

(The mean magnitude is taken as 6·10.)

	(1).	(2).	(3).	(4).	(5).	(6).	(7).	(8).	(9).	(10).	(11).	(12).
Curve	+42	+34	+18	− 3	−20	−33	−39	−30	−17	0	+15	+34
$p-f$	0	+24	+37	+38	+30	+19	− 3	−22	−30	−32	−34	−27

3. In the discussion of corrections to epoch, ANB's curve was used; but it will be seen from his residuals in Table III. that the differences are too small to make any great change probable, when the mean curve is substituted. There are, however, very sensible *constant* differences between observers, as shown in the third column of Table II. (The other columns will be explained presently.)

TABLE II.

Systematic Differences of Observers from Mean.

Weight.	Observer.	Mean Mag.	Scale.	Epoch.
11	A. N. Brown	+0·02	−·03	−·03 = −0·6 days.
9	A. A. Nijland	−0·12	+·06	+·07 = +1·4 ,,
5	F. de Roy	+0·38	+·08	+·02 = +0·4 ,,
4	G. B. Lacchini	−0·35	−·08	·00 = 0·0 ,,
1	C. F. Butterworth	−0·21	+·06	−·16 = −3·2 ,,
1	E. E. Markwick	+0·50	−·25	−·15 = −3·0 ,,

4. When these are applied with reversed sign as corrections, the individual deviations from the mean light curve are as below:—

TABLE III.

Residual Differences of Observers from Mean Light Curve in Units of 0·01 *mag.*

(1).	(2).	(3).	(4).	(5).	(6).	(7).	(8).	(9).	(10).	(11).	(12).
− 2	− 2	− 2	+1	+2	0	−3	+ 1	+4	+ 5	+ 3	−3
+ 3	+ 6	+ 6	−4	−2	0	+3	− 1	−5	− 8	− 4	+3
0	+ 1	+ 1	+6	−1	−4	−4	− 2	−1	− 3	+ 2	+7
+ 3	− 4	− 4	−1	+1	+5	+5	+ 3	+2	− 4	− 6	−1
+ 1	− 4	− 8	−1	+1	−6	−5	− 9	+1	+23	+11	−1
−15	−14	−10	+2	+1	+5	+4	+11	+9	+ 7	+ 7	−6

5. Inspection of these figures suggests that there are two classes of systematic difference, in addition to that in mean mag.

Firstly, some observers see the star brighter at maximum and fainter at minimum than others: F de R, for instance, makes the range sensibly greater than does EEM. The general test is an accordance in sign between the figures for the mean light curve and the residuals for the observer. If we multiply the two sets together and divide by the sums of squares of light curve, we get the numbers shown under the heading "Scale" in Table II. The significance of the quantity $-\cdot25$ for EEM is that the figures for the mean light curve must be diminished by one quarter to compare with his observations. His residuals

$$-15\ -14\ -10\ +2\ +1\ +5\ \ +4\ +11\ +9\ +7\ \ +7\ -6$$

would thus receive the corrections

$$+11\ \ +9\ \ +5\ -1\ -5\ -8\ -10\ \ -8\ -4\ \ 0\ \ +4\ +9,$$

making them

$$-4\ \ -5\ \ -5\ +1\ -4\ -3\ \ -6\ \ +3\ +5\ +7\ +11\ +3.$$

6. But this brings into clear view the other type of correction: viz. when the star is waxing (figures of light curve decreasing) EEM's corrected residuals are negative: *i.e.* he sees it rather brighter than the mean observer, the contrary when the star is waning. The personality is probably purely unconscious, but if it were conscious we might call EEM an eager observer, who hurries up the changes in the star. To assess his coefficient we may use the quantities $p-f$ of Table I. in a manner similar to that adopted for the light curve itself. We thus obtain the fifth column of Table II., a set of factors which, when multiplied by the values of $(p-f)$ given in Table I., give the appropriate corrections. But in more familiar form, this error assigns the number of days by which the whole curve is shifted to right or left; and this conversion is made in the last column of Table II. The quantities are insensible except perhaps for CFB and EEM; and in both these cases the paucity of observations may account for the error.* We shall ignore this error in what follows, though it seemed desirable to investigate it before combining the corrections to epoch found by the separate observers.

7. The mean light curve given in Table I. can be represented by the following expression

$$38{\cdot}21\cos(\theta-359^\circ)+2{\cdot}26\cos(2\theta-39^\circ)+1{\cdot}83\cos(3\theta-0^\circ),$$

which, reduced to the standard form, becomes

$$2{\cdot}83\cos(\theta-180^\circ)+0{\cdot}17\cos(2\theta-41^\circ)+0{\cdot}14\cos(3\theta-183^\circ),$$

which as it stands is a good example of Phillips's Group I., though the smallness of the coefficients makes the evidence somewhat uncertain.

* CFB's observations only extend over the last three cycles, while EEM's are blank from cycle 9 to 26.

8. Using, then, ANB's light curve, the next step was to determine the corrections to epoch for successive cycles as shown in Table IV.

TABLE IV.

Corrections to Epoch for Successive Cycles of W Cygni.
Adopted Period 129·33 Days.

Cycle.	Adopted Max.	ANB.	AAN.	F de R.	EEM.	GBL.	CFB.	Total Mean.	ϕ.
1	07 July 15·3	(+ 8)	...	+ 5	...	...	...	+ 6	96
2	07 Nov. 21·7	*−29*	...	−18	(−17)	...	...	−24	288
3	08 Mar. 30·0	+16	+ 6	...	...	...	...	+12	120
4	08 Aug. 6·3	− 9	(−12)	(0)	(−12)	...	...	− 8	312
5	08 Dec. 13·7	+19	+14	+14	(+ 1)	...	...	+14	144
6	09 Apr. 22·0	(+ 1)	−14	(−13)	...	...	...	−13	336
7	09 Aug. 29·3	− *8*	− 8	−11	(−16)	...	...	−10	168
8	10 Jan. 5·7	− 4	− 8	...	...	...	...	− 6	0
9	10 May 15·0	...	−10	...	...	...	...	−10	192
10	10 Sept. 21·3	+14	(+ 3)	...	...	...	...	+ 9	24
11	11 Jan. 28·7	−32	−30	...	...	...	...	−31	216
12	11 June 7·0	(0)	+ 3	...	...	...	...	+ 2	48
13	11 Oct. 14·3	...	−12	...	...	...	...	−12	240
14	12 Feb. 20·7	...	− 7	...	...	...	...	− 7	72
15	12 June 29·0	−10	*−17*	...	...	−11	...	−13	264
16	12 Nov. 3·3	+ 3	+ 3	...	...	+10	...	+ 6	96
17	13 Mar. 14·7	...	−10	...	...	− 9	...	− 9	288
18	13 July 22·0	+ 3	− 4	...	...	...	...	0	120
19	13 Nov. 28·3	(+ 2)	− 6	...	...	− 5	...	− 4	312
20	14 Apr. 6·7	(+ 3)	(− 5)	...	...	(− 4)	...	− 3	144
21	14 Aug. 14·0	−16	−18	...	...	*−10*	...	−13	336
22	14 Dec. 21·3	(−10)	0	+ 9	...	− 2	...	0	168
23	15 Apr. 29·7	...	+ 2	...	...	(−11)	...	− 4	0
24	15 Sept. 6·0	(− 4)	+ 5	*+16*	...	0	...	+ 8	192
25	16 Jan. 13·3	(+ 2)	+ 5	+12	...	(+11)	...	+ 9	24
26	16 May 21·7	+38	+34	+26	...	+38	...	+34	216
27	16 Sept. 28·0	+36	+39	+30	(+46)	+37	...	+36	48
28	17 Feb. 4·3	+37	(+37)	+30	...	+37	+36	+36	240
29	17 June 13·7	+59	+60	*+63*	...	*+54*	*+58*	+59	72
30	17 Oct. 21·0	(+37)	(+35)	(+38)	(+31)	(+40)	(+46)	+39	264

Italics represent specially good determinations. Figures in brackets poor determinations.

The most striking feature is the undoubted jump between cycles 25 and 26, which persists afterwards. There are differences of similar amount in other cases, as, for instance, between cycles 2 and 3, 5 and 6, or 10 and 11 ; but they do not persist afterwards. At cycle 25 there is apparently a permanent change of epoch, confirmed by five observers in good accord. Such unimpeachable

evidence has a value beyond the particular instance, in supporting cases in other stars where a sudden change of epoch is suggested, without, however, the clearness here manifested.

9. But there are other changes of epoch, too large and too well confirmed by more than one observer to be treated as merely accidental; and moreover there seems to be a systematic character about the residuals up to cycle 25. The + and − signs are nearly alternate, and there is a tendency to repeat after a number of cycles, which was in the first instance estimated at 19. The start was, moreover, false in taking 9 repetitions in the 19 cycles, giving an advance of rather *less* than 180° per cycle (actually $9 \times 360°/19 = 170°{\cdot}5$), whereas the true advance is rather *more* (thus $10 \times 360°/19 = 189°{\cdot}5$ was found to be an improvement). Ultimately the advance was fixed at 192° per cycle, so that in 15 cycles of θ the new term ϕ runs through its period eight times. All this work by trial and error took a considerable time, the whole of the observations being rearranged more than once; but the ultimate result justified the labour, since when the advance of 192° had been identified the observations gave a remarkably smooth curve as shown in Table VII., where the means for all six observers are grouped for every 8° of the new argument ϕ.

10. But before proceeding to this detailed analysis, let us arrange the corrections to epoch shown in Table IV. according to the argument ϕ shown in the last column. They stand as in Table V. We may remark that the period of ϕ is $15 \times 129{\cdot}33/8$ days $= 242{\cdot}5$ days, so that ϕ advances by $1°{\cdot}49$ per day, or 72° in 48·5 days.

TABLE V.

Corrections to Epoch in Table IV. according to $(\phi - 96°)$.

ϕ.	Corrections to Epoch.	Mean.	ϕ.	Corrections to Epoch.	Mean.	ϕ.	Corrections to Epoch.	Mean.
°			°			°		
0	+ 6 +6	+6	120	−31 (+34)	−23	240	−13 −13	−13
24	+12 0	+6	144	−12 (+36)	−12	264	− 6 − 4	− 5
48	+14 −3	+6	168	−13 (+39)	−11	288	+ 9 + 9	+ 9
72	−10 0	−5	192	−24 − 9	−16	312	2 (−36)	5
96	−10 +8	−1	216	− 8 − 4	− 6	336	− 7 (+59)	+ 2

11. The numbers in brackets represent the cycles 26 to 30; they obviously exceed the corresponding numbers in previous cycles by about 50 days. Table VI. shows the excess in order of date; cycles 23, 24, 25 being added to show the break.

TABLE VI.

Cycle . .	23.	24.	25.	26.	27.	28.	29.	30.
ϕ	0	192	24	216	48	240	72	264
Excess	+2	+18	0	+65	+34	+48	+66	+52

There is apparently a sudden jump between cycles 25 and 26. The mean excess for cycles 26 to 30 is +53 days, while for the 5 preceding cycles it is +6 days.

12. Adopting $-48^{\cdot}5$ days (equivalent to 72° in ϕ) as the correction to be applied to the bracketed numbers in Table V., and then taking means of the pairs of results, we get the numbers shown in the third column of Table V., which, when analysed harmonically, give—

$$10^{\cdot}1\cos(\phi-353^\circ)+2^{\cdot}9\cos(2\phi-46^\circ)+2^{\cdot}6\cos(3\phi-145^\circ).$$

13. But having now this clear evidence of the existence of a term of $242^{\cdot}5$ days, it is better to go back to the original observations and correct them for ϕ, rather than to pursue further the study of these corrections to epoch obtained on the faulty supposition of a simple period. The results for all the observers were combined to form a series of mean values at values of ϕ differing by 8°, giving 45 terms to the cycle of ϕ; these are shown in Table VII., column O, and are remarkably smooth and consistent.

On analysing the observed values O harmonically the following expression was obtained, the unit being $0^{\cdot}001$ magnitude—

$$256\cos(\phi-4^\circ)+18\cos(2\phi-238^\circ)+45\cos(3\phi-270^\circ).$$

Regarding the terms after the first as unreal, the column C in Table VII. represents the first term only. The residuals O − C are very satisfactory, 31 out of the 45 being comprised between the limits ±3—that is, between $\pm{\cdot}03$ magnitude.

TABLE VII.

Means according to Value of ϕ, *which advances* 192° *in each adopted Period of* $129^{\cdot}33$ *Days, thus completing its Cycle in* $242^{\cdot}5$ *Days.*

(The unit is $0^{\cdot}01$ magnitude.)

ϕ.	O.	C.	O−C.	ϕ.	O.	C.	O−C.	ϕ.	O.	C.	O−C.
°				°				°			
0	+29	+26	+ 3	120	− 3	−11	+8	240	−21	−14	− 7
8	+23	+26	− 3	128	−17	−14	−3	248	−13	−11	− 2
16	+30	+25	+ 5	136	−20	−17	−3	256	− 2	− 8	+ 6
24	+25	+24	+ 1	144	−22	−20	−2	264	− 3	− 4	+ 1
32	+20	+23	− 3	152	−25	−22	−3	272	0	− 1	+ 1
40	+ 8	+21	−13	160	−24	−23	−1	280	0	+ 3	− 3
48	8	+18	−10	168	−22	−25	+3	288	− 6	− 6	0
56	+13	+16	− 3	176	−22	−25	+3	296	+ 8	+10	− 2
64	+15	+13	+ 2	184	−23	−26	+3	304	+13	+13	0
72	+19	+10	+ 9	192	−30	−25	−5	312	+22	+16	+ 6
80	+ 8	+ 6	+ 2	200	−29	−25	−4	320	+29	+18	+11
88	0	+ 3	− 3	208	−25	−23	−2	328	+22	+21	+ 1
96	+ 1	− 1	+ 2	216	−17	−22	+5	336	+23	+23	0
104	+ 2	− 4	+ 6	224	−23	−20	−3	344	+16	+24	− 8
112	− 6	− 8	+ 2	232	−18	−17	−1	352	+25	+25	0

14. The values C, with the constant 610 (or 6·10 magnitudes), were then subtracted from the individual means; and these were then arranged in θcycles of 129·33 days, twelve portions to a cycle, as in Table VIII. For the cycles 25 to 30 the crude hypothesis was made that the variation halted for 24 days before commencing cycle 25, again for 12 days in the middle of cycle 26, and again for 12 days early in cycle 28, making 48 days' halt in all.

TABLE VIII.

Means of 6 *Observers arranged according to* θ *after subtracting* $610 + 26\ cos\ (\phi - 0°)$.

(The unit is 0·01 magnitude, and the Julian date corresponding to the first term is 2417704.)

Cycle.	(1).	(2).	(3).	(4).	(5).	(6).	(7).	(8).	(9).	(10).	(11).	(12).
1	+20	...	...	− 5	−19	−24	− 11	+ 1	+ 6	+10	+14	+31
2	+12	− 3	−12	−25	−35	−18	− 5	+ 4	+15	+23	+28	+33
3	+24	...	...	...	−34	−47	− 41	−27	−10	0	+ 3	+12
4	+27	+19	+ 7	− 7	−15	−26	− 47	−34	−16	+ 4	+21	+44
5	+41	+36	+23	+ 1	−30	−63	− 42	−27	−17	+ 3	+33	+26
6	+ 7	...	−11	...	− 4	− 7	− 11	− 4	+13	+23	+27	+36
7	+20	+ 1	−16	−35	−48	−44	− 39	−28	− 2	+24	+44	+38
8	+35	+27	+14	+ 2	−17	−11	− 5	− 3	+ 1	+14	+32	+47
9	+67	+44	+18	− 6	−22	−28	− 29	+ 4	+27	+45	+69	+61
10	+50	+41	+37	+17	−24	0	+ 11	+16	+15	+19	+30	+25
11	+21	+ 7	−10	−27	−37	...	− 10	+14	+46	+70	+52	+41
12	+39	+22	− 6	−38	−49	−45	− 36	−25	− 8	+10	+20	+24
13	+31	− 5	...	...	−32	−35	− 37	−36	−26	−12	+ 2	+14
14	+28	+34	−28	−64	−77	−39	− 31	−20	+ 3	+24	+54	+56
15	+56	+20	+ 1	− 9	−24	−46	− 36	−29	−15	+11	+35	+52
16	+66	+33	− 4	−32	−45	−51	− 43	−29	− 9	+ 6	+28	+62
17	+31	+ 5	− 8	−18	−32	−50	− 67	...	−46	−26	− 1	+13
18	+16	− 5	− 8	−30	−52	−66	− 38	−18	+ 4	+30	+45	+48
19	+49	+29	+ 5	−15	−38	−73	−106	−79	−48	− 8	+45	+29
20	+13	+ 2	− 9	−34	...	−53	− 39	−20	+ 3	+23	+37	+51
21	+46	+31	0	31	−49	−56	− 40	−17	0	+15	+32	+51
22	+47	+38	+22	− 1	−20	−29	− 21	− 9	0	+ 5	+13	+39
23	+27	...	+ 3	...	−19	−23	− 18	−13	−10	− 9	− 1	+ 7
24	+16	+30	+20	+13	+ 7	+ 2	+ 6	+ 4	+ 7	+20	+31	+39
25	+62	+41	+13	−12	−24	−34	− 24	−12	+ 1	+16	+29	+44
26	+42	+35	+16	−10	−23	−26	− 22	−19	−13	+ 2	+15	+24
27	+37	+26	+ 2	−28	−59	−83	− 77	−59	−29	...	+18	+26
28	+13	+ 4	− 1	−14	−35	−49	− 54	−41	−17	+ 5	+24	+38
29	+34	+29	+20	+ 7	− 7	−18	− 24	−27	− 9	+11	+30	+39
30	+39	+20	+ 6	− 4	−18	−38	− 34	−28	−19			
Mean	+32	+21	+ 4	−15	−28	−35	− 33	−22	− 7	+10	+25	+35

15. Forming the first harmonic for each cycle, the coefficients range closely about a mean value 38 (*i.e.* 0·38 magnitude) and the phases about the mean value 336°. Table IX. shows the differences from these means. There is a slow increase in the phase, and the second set of differences gives a comparison with values which allow for this increase, as explained below.

TABLE IX.

Analysis of the 30 Cycles according to θ Residuals from Mean Coefficient 38 and Mean Phase 336°.

Cycle.	Coeff.	Phase.		Cycle.	Coeff.	Phase.		Cycle.	Coeff.	Phase.	
		°	°			°	°			°	°
1	−17	− 9	+ 2	11	+ 8	−46	−43	21	+12	− 5	− 9
2	− 9	−41	−31	12	+ 2	− 5	− 2	22	− 5	+16	+11
3	− 8	+ 1	+10	13	−10	+10	+12	23	−19	+23	+17
4	− 3	+16	+25	14	+16	−21	−20	24	−24	+ 9	+ 3
5	+ 7	+17	+25	15	+ 7	+ 6	+ 6	25	+ 3	+ 6	− 1
6	−20	−29	−22	16	+16	+ 2	+ 2	26	− 7	+19	+11
7	+ 6	−18	−12	17	+ 2	+27	+26	27	+20	+11	+ 2
8	−12	+ 3	+ 9	18	+11	−21	−19	28	+ 1	+ 5	− 4
9	+11	− 9	− 4	19	+30	+20	+17	29	− 6	+21	+11
10	−17	+10	+14	20	+ 7	−19	−22	30	− 1	+11	0
Mean	− 6	− 6	+ 2	Mean	+ 9	− 5	− 4	...	− 3	+12	+ 4

16. The means of the 3 groups of 10 shown at the foot of the table show the consistency of the results and also the gradual change of phase. Means for successive groups of 5 are—

Coeff.	− 6	−6	+ 5	+13	− 7	+ 1
Phase	− 3°	−9°	−11°	+ 2°	+10°	+13°
Calc.	−10	−6	− 2	+ 2	+ 5	+ 9

As regards the progressive change of phase, we must remember that the last group is affected by the allowance made for the discontinuity about cycles 25–28; but we can determine the rate of change of θ from the first five groups alone as

$$\frac{20 \times 13^\circ + 10 \times 11^\circ}{20^2 + 10^2} = 0^\circ{\cdot}74 \text{ per cycle,}$$

giving calculated values as shown; from which it appears that the allowance made for the discontinuity cannot be far wrong. The period corresponding to this corrected value of θ is

$$129{\cdot}33 \Big/ \left(1 - \frac{0^\circ{\cdot}74}{360^\circ}\right) = 129{\cdot}60 \text{ days.}$$

17. The second and third harmonics of θ were also calculated: but their phases show no accordance through different cycles. They are distributed all round the circuit as below.

0°–59°.	60°–119°.	120°–179°.	180°–239°.	240°–299°.	300°–359°.
4	6	5	3	8	4
2	4	6	9	6	3

For the present we must regard these terms as unsubstantiated.

18. To obtain the best value for the coefficient of the first harmonic, we form the mean value of $R\cos(\psi-\psi_0)$, where R is the observed coefficient for any cycle, ψ the observed phase, and ψ_0 the calculated phase, assuming the progressive change of 0°·74 per cycle. The mean comes out 36, or 0·36 magnitude.

19. It seems clear, then, that the formula

$$6{\cdot}10+0{\cdot}26\cos(\phi-0^\circ)+0{\cdot}36\cos(\theta-336^\circ)$$

represents the observations at any rate to a good first approximation, though it may not be the complete representation. For investigating additional possibilities it will be well to clear these main terms out of the way, adopting the revised values of the periods of θ and ϕ, viz. 129·60 days for θ and 243·0 days for ϕ. This has been done in Table X., which gives residuals for the mean of six observers for every 10·8 days. Twelve of these time intervals make up the θ cycle, and are arranged in twelve columns. The ϕ cycle contains $22\frac{1}{2}$ of them; or, in other words, the ϕ cycle advances 192° for each cycle of θ. Thus, in every column θ is constant (or changes by 360° each time), while ϕ changes by 192°, and it is easy to test for additional terms by analysing each column in terms of ϕ. But before doing so, we see almost by inspection of Table X. that there is some long-period term which makes a whole row positive about the ninth, and again about the twenty-fourth row or θ cycle.

20. This suggests a term running through its period in 15 cycles of θ, or 8 of ϕ; we may denote the new variable by χ, but we have the relation

$$\chi=2\phi-\theta.$$

Further, other considerations raised the question whether perhaps there is a term in $\theta/2$ running independently.

21. On arranging the residuals of Table X. in 15 groups of 24 terms, reduced to 12 for convenience by taking means of consecutive pairs, we find the mean values O, from which the term C is calculated from the formula

$$-4\sin\theta/2+2\cos\theta/2=5\cos(\theta/2-296^\circ),$$

O	+2	0	−2	−1	−2	−2	−4	0	+3	+10	+5	+4
C	+2	0	−2	−4	−4	−4	−2	0	+2	+4	+4	+4
O−C	0	0	0	+3	+2	+2	−2	0	+1	+6	+1	0

The residuals O − C suggest a small error to the term in θ, as well as a slight error in mean value; such as

$$+2-2\cos\theta.$$

TABLE X.

Means of Six Observers for every 10·8 *Days, after subtracting* $610 + 36\cos(\theta - 336°) + 26\cos(\phi - 0°)$, *in Units of* 0·01 *magnitude.*

(Cycles for θ and ϕ are 129·6od. and 243·0d. Initial date J.D. 2417704.)

	(1).	(2).	(3).	(4).	(5).	(6).	(7).	(8).	(9).	(10).	(11).	(12).
1	−11	− 7	+ 3	+14	+21	+22	+26	+21	+10	− 2	−10	−11
2	−16	−16	−13	− 7	− 6	+11	+19	+17	+12	+ 3	0	− 3
3	− 6	− 7	− 3	+ 2	+ 4	− 1	− 5	− 6	−13	−22	−26	−23
4	−11	− 1	+ 5	+12	+14	+10	− 5	−18	−17	−17	−12	0
5	+15	+20	+14	+12	+ 6	− 7	−13	−10	−12	−17	−17	−16
6	−22	−20	−12	− 1	+10	+20	+26	+24	+16	+ 6	+ 2	− 3
7	− 6	−10	−10	− 7	−10	− 8	0	+ 4	+ 3	0	+ 1	+ 4
8	+ 2	+ 3	+ 9	+18	+22	+21	+19	+19	+16	+ 9	+ 9	+14
9	+31	+26	+23	+23	+17	+ 6	+17	+23	+24	+25	+31	+26
10	+17	+14	+15	+29	+17	+30	+36	+33	+24	+13	+ 5	− 2
11	− 7	−11	−10	− 6	− 7	+ 6	+23	+33	+36	+35	+25	+ 6
12	− 9	−14	−15	−13	−17	− 4	+ 5	+ 3	− 3	−10	−13	− 9
13	− 9	−12	−10	− 6	− 5	− 5	− 8	−13	−18	−26	−29	−24
14	− 9	− 4	−14	−21	−30	− 7	+ 7	+11	+10	+ 8	+11	+19
15	+12	+ 6	+ 5	+ 5	+ 2	− 8	− 6	0	+ 2	+ 1	+ 4	+14
16	+22	+ 7	− 2	− 8	−17	−14	−10	− 4	− 3	− 3	+ 1	+12
17	− 2	− 6	− 6	− 4	− 7	−14	−33	−33	−34	−37	−34	−25
18	−15	−15	−16	−14	−15	−17	− 2	+ 7	+10	+10	+16	+14
19	+15	+ 6	− 1	− 5	−14	−28	−49	−60	−37	−15	+11	− 6
20	−18	−21	−19	−15	−15	−16	− 1	+ 6	+ 7	+ 6	+ 9	+14
21	+11	− 1	− 7	−11	−19	−18	− 3	+ 3	+ 4	+ 3	+ 7	+18
22	+19	+16	+16	+17	+14	+ 8	+15	+12	+ 6	− 2	− 6	− 2
23	+ 1	0	+ 4	+ 9	+10	+12	+10	+10	+ 2	− 1	+ 5	+ 9
24	+21	+24	+29	+36	+36	+32	+31	+32	+27	+21	+19	+19
25	+19	+12	+ 7	+ 4	+ 2	+ 9	+16	+16	+18	+17	+15	− 4
26	−18	−19	−14	− 4	+ 8	+17	+20	+16	+ 9	0	− 4	− 2
27	− 5	−21	−32	−41	−43	−19	− 5	+ 1	+ 3	+ 2	− 9	−18
28	−20	−15	−10	− 6	− 8	−16	−18	−11	− 8	− 7	− 1	+ 4
29	+ 1	+ 2	+11	+19	+24	+19	+ 6	+ 2	+ 1	− 2	+ 1	− 2
30	− 3	− 3	+ 1	+ 7	+ 8	+ 4	+ 3	− 3				

These terms are counted from the epoch midway between the first two terms of Table X., and need a reduction to the initial date. But for the present exploring work we need not dwell on this point. We see at once, from the above figures O, that the means of alternate rows of Table X. differ systematically: the odd rows have mean value − 1 (mean of first six terms in line O) and the even rows mean value + 3 (mean of last six).

22. It will be just as well to apply these corrections before studying the long-period term

$$\chi = (2\phi - \theta)$$

from the means of each row. Accordingly in Table XI. the means of the rows are given corrected by $+1$ and -3 alternately. Since χ repeats after 15 cycles of θ, the two sets are shown under the heads I. and II.

TABLE XI.

Means of rows of Table X. corrected for term in $\theta/2$.

Cycle.	I.	II.	II.−I.	Corrected.		C_1.	C_2.
1	+ 7	− 5	−12	+ 4	− 5	− 9	− 2
2	− 3	−19	−16	− 6	−19	− 9	− 7
3	− 8	− 6	+ 2	−10	− 5	− 8	−12
4	− 6	−14	− 8	− 8	−13	− 6	−12
5	− 1	− 8	− 7	− 3	− 7	− 2	− 6
6	+ 1	0	− 1	− 1	+ 1	+ 2	0
7	− 2	+ 6	+ 8	− 4	+ 7	+ 6	+ 6
8	+10	+ 7	− 3	+ 9	+ 9	+ 8	+12
9	+24	+25	+ 1	+23	+27	+10	+18
10	+16	+14	− 2	+15	+16	+10	+18
11	+11	− 2	− 9	+10	0	+ 7	+ 3
12	−11	−15	− 4	−12	−13	+ 4	−12
13	−13	−13	0	−13	−10	0	−12
14	− 5	+ 6	+11	− 5	+ 9	− 4	− 4
15	+ 4	− 1	− 5	+ 4	+ 2	− 7	+ 3

23. When we take the difference II.−I. we see that it is usually negative and has a mean value -3. In the columns headed "Corrected" this has been allowed for as a steady fall in mean value, one unit every five terms. The mean of the two columns was then analysed harmonically (for a single term in χ), the numbers C_1 being the result; but it will be seen that these are inadequate to explain all the features of the columns. Other harmonics are needed. To test their reality we may analyse the two columns separately, obtaining the figures below:—

Set I. $8\cos(\chi - 190^\circ) + 9\cos(2\chi - 22^\circ) + 6\cos(3\chi - 289^\circ)$
Set. II. $8\cos(\chi - 195^\circ) + 9\cos(2\chi - 345^\circ) + 7\cos(3\chi - 251^\circ)$.

These are closely consistent and are good evidence of deviation from a simple harmonic term, since the first coefficient is no larger than others. We can represent the residuals by four straight lines as shown in column C_2, the secondary maximum being clearly shown.

24. We may now return to the possibility of analysing each column of Table X. for ϕ. The question arises whether we should analyse them as they stand, or first correct them for the terms in χ and $\theta/2$ already found. Both methods were tried. It may be

well to consider, before presenting the results, one possible origin for a term in θ and ϕ conjointly.

25. The so-called "light curve" of a variable is not really a light curve, but a magnitude curve representing logarithms of light. Suppose that the fluctuations represented by θ and ϕ are really fluctuations in *brightness*, which occur independently, so that the brightness B is

$$B = B_0(1 + a\cos\theta' + b\cos\phi'),$$

the dashes to θ and ϕ denoting that the zero points are to be determined later. The magnitude m is deduced by taking logarithms; and it will be convenient to choose e as base rather than 10, so that the magnitude ratio is $2{\cdot}30 \times 0{\cdot}4 = 0{\cdot}92$. Then

$$\begin{aligned} C - 0{\cdot}92m &= \log B_0 + \log(1 + a\cos\theta' + b\cos\phi') \\ &= \log B_0 + (a\cos\theta' + b\cos\phi') - \tfrac{1}{2}(a\cos\theta' + b\cos\phi')^2 + \text{etc.} \end{aligned}$$

But now we have found that the first-order terms in m are

$$m = 6{\cdot}10 + 0{\cdot}36\cos(\theta - 336^\circ) + 0{\cdot}26\cos\phi,$$

so that

$$-0{\cdot}92m = -5{\cdot}61 + 0{\cdot}33\cos(\theta - 156^\circ) + 0{\cdot}24\cos(\phi - 180^\circ).$$

Hence we have

$$\begin{aligned} a\cos\theta' &= 0{\cdot}33\cos(\theta - 156^\circ), \\ b\cos\phi' &= 0{\cdot}24\cos(\phi - 180^\circ). \end{aligned}$$

The second-order terms in m are therefore

$$0{\cdot}059\cos^2\theta' + 0{\cdot}086\cos\theta'\cos\phi' + 0{\cdot}031\cos^2\phi'.$$

The square terms are equivalent to

$$0{\cdot}030\cos 2\theta' + 0{\cdot}016\cos 2\phi' + \text{const.}$$

The coefficients are becoming small, and such terms might be masked by accidental errors: see § 14 for $2\phi'$ and § 17 for $2\theta'$.

26. But the product term, which is equal to

$$0{\cdot}043[\cos(\theta' - \phi') + \cos(\theta' + \phi')],$$

seems to offer a better chance of detection. The periodic terms increase by $30^\circ \pm 16^\circ$ for each term of Table X., completing their periods in 278 days and 84·5 days; or in 25·7 and 7·83 terms of Table X. Arranging the residuals first in sets of 7·83 terms (counted as 8 with one term repeated in every 6×8), the means come out

$$+{\cdot}020 \quad +{\cdot}013 \quad +{\cdot}003 \quad -{\cdot}003 \quad +{\cdot}005 \quad +{\cdot}011 \quad +{\cdot}009 \quad +{\cdot}019,$$

or, correcting by the constant $-{\cdot}010$, we get

$$+{\cdot}010 \quad +{\cdot}003 \quad -{\cdot}007 \quad -{\cdot}013 \quad -{\cdot}005 \quad +{\cdot}001 \quad -{\cdot}001 \quad +{\cdot}009.$$

There might thus be a small term of coefficient ·010, but there can be no term of size ·043. The means are unusually small, and suggest that terms which show any coefficient of ·030 or over are probably real.

27. Again arranging in 25·7 days (counted as 26 with a term repeated every three sets), and taking means of consecutive pairs for convenience, the means of columns were analysed and gave the terms

$$+\cdot008 \;+\cdot016\sin\psi \;-\cdot005\cos\psi \;-\cdot001\sin 2\psi \;+\cdot039\cos 2\psi,$$

with residuals, in units of ·001 magnitude,

$$-15 \;+2 \;+7 \;+4 \;-12 \;+2 \quad 0 \;+11 \;-11 \;-14 \;-5 \;+27 \;-6.$$

28. There is clearly no first harmonic with coefficient as large as ·043. The second harmonic was obvious by inspection, and its coefficient is large enough to suggest reality. The letter ψ has been written for $(\theta' - \phi')$, so that

$$2\psi = 2(\theta' - \phi') = 2(\theta - \phi) - 312^\circ,$$

and the period is 139 days, which is near enough to the main period of 129·6 days to cause sensible interference.

29. But, as already remarked, product terms like $\cos\theta' \cos\phi'$ may be considered in another manner; since in each column of Table X. θ' is constant, we may analyse the columns for ϕ. The value of ϕ for the first term of the first column is 4°, for the first term of the second column 20°, of the third column 36°, and so on; but it was more convenient in practice to ignore these differences until the analysis had been performed, applying them to the deduced phase at the end. The first fifteen rows give a complete cycle of ϕ called first set, and the rows 16–30 give a second set. The analysis was conducted for each set independently.

TABLE XII.

Analysis of Separate Columns of Table X.

Col.	·035 cos (θ − 156°).	Whole Series.		First Set.		Second Set.		Phase. II. − I.
		Coeff.	Phase.	Coeff.	Phase.	Coeff.	Phase.	
			°		°		°	°
(1)	− ·032	·02	10	·06	77	·07	295	218
(2)	− ·021	·02	68	·05	74	·01	284	210
(3)	− ·004	·02	133	·02	92	·03	156	64
(4)	+ ·014	·03	176	·02	320	·08	168	208
(5)	+ ·028	·04	199	·04	304	·10	181	237
(6)	+ ·035	·02	317	·10	347	·07	187	200
(7)	+ ·032	·05	101	·09	13	·13	143	130
(8)	+ ·021	·07	136	·04	40	·16	184	144
(9)	+ ·004	·05	158	·02	5	·13	162	157
(10)	− ·014	·03	179	·03	313	·09	166	213
(11)	− ·028	·01	220	·04	324	·05	170	206
(12)	− ·035	·01	246	·04	23	·06	216	193

30. If there is a term in $\cos\theta' \cos\phi'$, then the coefficients deduced for ϕ should change sign for columns (4) to (9), as shown by the second column; or, rather, the phases deduced for ϕ should show a *per saltum* change of 180°. Omitting columns (3) and (9), for which the theoretical coefficient is small, let us change the phases for columns (4) to (8) by 180°. They then stand as follows:—

(1)	10	(4)	356
(2)	68	(5)	19
(10)	179	(6)	137
(11)	220	(7)	281
(12)	246	(8)	316

31. There is clearly no consensus here. Let us now consider the two sets separately. Firstly, we see that their coefficients are larger than that for the whole series, and secondly, the column II. – I. shows us that the phases differ by about 180°. Thus, terms such as $\cos\phi \sin\theta/2$ (which changed sign in alternate rows with the advance of $\theta/2$ by 180°) are suggested as more likely than terms such as $\cos\phi \cos\theta$, which have the same sign for all rows.

32. To test for the existence of such terms, the χ and $\theta/2$ terms were first cleared from Table X., and the columns were analysed afresh. They were also divided in a rather different way, not according to date (early and late) but according to value of $\theta/2$. All the odd rows were taken together to form a cycle in ϕ; and all the even rows were analysed separately to form another cycle. For terms which are the same in ϕ, but have a factor $\cos\theta/2$ or $\sin\theta/2$, the two sets will simply differ in sign; and the coefficients of $\sin\phi$ and $\cos\phi$, calculated for the twelve columns, duplicated by the odd and even rows, should each give 24 terms following a cycle in $\theta/2$.

33. In this way the following expression was derived to represent the residuals—

$$\begin{aligned}&\sin\phi\,[+0\cdot015 + 0\cdot035\sin\theta/2 + 0\cdot015\cos\theta/2]\\ &+\cos\phi\,[-0\cdot024 - 0\cdot009\sin\theta/2 - 0\cdot017\cos\theta/2].\end{aligned}$$

The terms $0\cdot015\sin\phi - 0\cdot024\cos\phi$ must be taken as small corrections to the adopted ϕ term. The others may be put into the form

$$+\cdot026\cos(\phi + \theta/2 - 173^\circ) + \cdot015\cos(\phi - \theta/2 - 53^\circ).$$

It seems doubtful whether such small terms are significant, though they show a distinct persistence when treated in different ways.

34. The important point, from the physical point of view, is that we get no evidence of terms which might arise in the manner suggested in § 25 from the conversion of brightness into magnitude. It almost seems as though the magnitude curve may be a closer representation of the physical phenomena than the brightness curve. Is this so in general?

35. As there seemed to be several possible periodicities in addition to those of θ and ϕ, it was ultimately decided to conduct a systematic search by the method of the periodogram, or rather of the Fourier Sequence. The intensities and phases of all the Fourier terms up to divisor 33 were calculated from the residuals of Table X., but it will be seen that since the reality of most of them is doubtful, it is superfluous to give the phases, and Table XIII. is limited therefore to the intensities (sums of squares of coefficients) as in the periodogram.

TABLE XIII.

Intensities for the Fourier Sequence in Units of 0·01 *magnitude (squared).*

Divisor.	Intensity.	Divisor.	Intensity.	Divisor.	Intensity.
1	7	13	0	25	13
2	75	14	2	26	8
3	7	15	29	27	0
4	73	16	655	28	13
5	1	17	32	29	18
6	49	18	6	30	1295
7	3	19	0	31	15
8	0	20	8	32	6
9	13	21	1	33	0
10	6	22	5	34	
11	1	23	9	35	
12	5	24	4	36	21

36. The compilation of this table involved a good deal of labour, but it was well expended, for it brings out the facts very clearly. In the first place, there is no doubt whatever about the two main periodicities θ and ϕ: their intensities 655 and 1295 are quite conspicuous.

Secondly, there are 21 intensities less than 10, of which the mean is 4. These are undoubtedly all accidental.

37. Thirdly, there are half a dozen ranging from 13 to 21, which are also probably accidental. If we include them with the second group, the mean accidental intensity rises to 7, and none of the present group exceeds three times this value. Among them are the 18 and 15 for the terms adjacent to θ; and experience prepares us for terms slightly above the average on the two sides of a marked periodicity. Thus meteorological elements show marked annual terms; and it is a common experience to find, in reducing any long series of observations, that the Fourier terms next to the exact year on each side of it are also large. The explanation is that any "accidental" deviation from the exact year, such as the annual maximum occurring a little earlier than usual or a little later, reinforces these adjacent terms. For this reason we may in all probability exclude as unreal the intensities 29 and 32 adjacent

to ϕ. We have above identified the former of these with $\theta/2$, which is a possible "real" term; but looking at the matter from this point of view, its reality is more doubtful, and for the present it seems better to regard it as spurious. It will be noticed that we have also rejected the terms in $\phi+\theta/2$, $\phi/2+\theta/2$, and $\phi-\theta/2$.

38. There remain now only the three terms in χ or $2\phi-\theta$. Their intensities 75, 73, and 49 are considerable multiples of the average 7, or even of a new average 9 obtained by including 29 and 32 as accidental. Secondly, the agreement of the two cycles for the χ terms, shown in § 23, is very good and suggests reality. Thirdly, if we clear the residuals of Table X. for these terms in χ, it is found that the distribution is markedly improved, though on consideration it was determined not to burden this paper with the tables showing the improvement. For the single repetition of the χ term is subject to the obvious suspicion attending all such terms with small divisors. The repetition of large residuals in Table X. is obvious to the eye, but will it recur in the future? It seems better to leave it to the future to decide.

39.* There is, however, one further point to be considered in connection with a possible physical interpretation of the double variation. Can the star be included with other long-period variables? The smallness of the range in variation is at first sight against this; but there is a possibility which is at present little more than speculation, but is worth considering. Phillips has divided the variables into two classes. It has been shown (*M.N.*, 79, 371) that in each class the length of period is a clue to certain ordered features suggesting a progression, or development, and that there is a possible point of junction of the two series where the periods are small. Now, whether we take the period 129 days or 243 days, the star would be assigned to this point of possible junction, and the question arises whether this star may not belong to *both* classes. Is it perhaps in the act of changing from one to the other? Is a variation of one type dying down and another one arising? The smallness of the range (less than one magnitude instead of several magnitudes) would then be naturally explained. Further, we *might* be able to see which type is waxing and which is waning, which would assign the order of passage from one type to another; though, of course, the changes might be too slow to be made manifest in a few years.

40. Looking at the material from this point of view, we find that the θ variation shows signs of waxing and the ϕ type of waning. In § 15 the coefficients for the θ variation are shown, and though it cannot be said that the evidence is conclusive, it all points one way. The mean coefficient for the first 10 cycles is $38-6=32$, and for cycles 16 to 25 is $38+3=41$. [It seems better to omit the cycles affected by the halt.] Turning now to Table XII., the variation of ϕ was determined from two sets of 15 cycles which differ by five years in date. There is a notable difference between

* From this point the paper ceases to be a joint production.—M. A. B.

the phases in the two sets which has been already remarked and tentatively ascribed to a term in $\theta/2$. But an alternative possibility is simply that the coefficient of ϕ variation is decreasing; for Table XII. assigns as corrections to the adopted value

	$0^{\cdot}00 \sin\phi + 0^{\cdot}26 \cos\phi$
from early observations	$+0^{\cdot}01 \sin\phi + 0^{\cdot}03 \cos\phi$,
from late observations	$+0^{\cdot}01 \sin\phi - 0^{\cdot}07 \cos\phi$,

making the coefficient change from $+0^{\cdot}29$ to $+0^{\cdot}19$ in five years. This diminution in ϕ variation, associated with an increase in θ variation, suggests that the star is changing from Phillips's Group II. to Group I.; for a comparison of Tables I. and VIII. of the paper above quoted will show how a period of 243 days fits in at the end of Group II. (which extends no lower than 177 days), and a period of 129·6 days at the end of Group I., though it is actually just outside the table. In § 7 the θ variation was referred to Group I., at a time when these possibilities of change were quite unsuspected, and indeed the ϕ variation had not been detected. In § 14 the ϕ variation is found to be

$$0^{\cdot}256 \cos(\phi - 4^{\circ}) + {\cdot}018 \cos(2\phi - 238^{\circ}) + {\cdot}045 \cos(3\phi - 270^{\circ}).$$

Here the coefficients are smaller still, and conclusions therefore yet more uncertain: indeed, the words "Regarding the terms after the first as probably unreal" have been left as written many months ago. But taking the figures for what they are worth, and throwing them into the form

$$1^{\cdot}00 \cos(\phi - 180^{\circ}) + 0^{\cdot}07 \cos(2\phi - 230^{\circ}) + 0^{\cdot}18 \cos(3\phi - 78^{\circ}),$$

we see that the type is that of Group II. The value $\phi_2 = 230^{\circ}$ is rather high for the end of Group II. (see *M.N.*, 79, 376, Table V.), but at any rate $\cos\phi_2$ is negative, which is taken as the character of Group II.

The values of ϕ_3, k'_2, and k'_3 fit as well as could be expected with such small quantities. We may thus say that all the observed facts are consistent with a crossover from Group II. to Group I.

41. The notion above suggested, of progressive change down Group II. from long periods to short, then across to Group I., and up from short periods to long, carries with it some corollaries which are capable of being tested independently of this star. For instance, the periods of Group II. ought to be shortening in general, and those of Group I. ought to be lengthening. Now, changes in period have long been recognised, and have hitherto been expressed by such formulæ as

$$239\,9197 + 308^{\cdot}3\,E + 35 \sin(9^{\circ}E + 86^{\circ}),$$

which is Chandler's formula for S Herculis (Ch. 6044, Harv. 164715). Do these changes give us any information on the point above raised? It is clear that, as they stand and if they are taken as accurate, they can tell us nothing about secular changes, being

themselves periodic. But they are well known to be provisional estimates awaiting confirmation; and as an outcome of prolonged scrutiny of the Baxendell stars, the present writers have found no case where strict periodicity has been confirmed (*M.N.*, 78, 539), in spite of expectation originally to the contrary. Thus, for the star S Herculis above mentioned, the Rev. T. E. R. Phillips was driven to abandon the hypothesis of a simple sine-term, and to substitute a series of five harmonic terms (*M.N.*, 75, 563). It was later suggested by the present writers (*M.N.*, 76, 185) that two simple changes of period would adequately represent the observations, thus:—

E =	1	13	48	70	
Error of Epoch =	+10	−65	+75	−35	days
Correction to Period	−6·25	+4·00	−5·00		,,
Period	299·95	310·20	301·20		,,

Now, there is here undoubtedly an oscillation in the period, which first lengthens and then diminishes; but the significant fact for us would appear to be that the return is not complete, the final value 301·20 days being sensibly larger than the original value 299·95 days. We thus have apparently a secular term accompanying the periodic one: the evidence, so far as it goes, shows the period to be lengthening. It is therefore satisfactory, from our present point of view, to see that S Herculis is in Phillips's Group I., having $\phi_2 = 283°$.

Moreover, the same reference gives us a Group II. star going the other way. The figures quoted for S Herculis were given in a discussion of R Leonis (Ch. 3493, Harv. 094211), which is a Group II. star ($\phi_2 = 114°$), and for it also two changes of period were suggested, viz. from 311·80 days to 314·44 and then back to 310·70, now *smaller* than at first. There is some uncertainty about these determinations, but they have the advantage of having been made and published before the above hypothesis was suggested, and are thus quite unbiassed. They make it clear that a single change of period may be inconclusive: we must wait for the return swing, and they thus enhance the value of observations which already extend over long periods, suggesting further scrutiny of the old observations in the light of this suggestion.

42. Looking through the other Baxendell stars recently discussed by the present writers, the following relevant cases are found. The Group is that assigned on the revised principle (according to the sign of $\cos \phi_2$) in *M.N.*, 79, 371, and sometimes differs from that provisionally given on p. 540 of *M.N.*, 78.

Group II.

χ Cygni: 409 days. No change? (*M.N.*, 78, 519).

U Herculis: 408 days. Decreasing with fluctuation:—410, 413, 403, 408 days (*M.N.*, 78, 526).

S Coronæ: 360 days. Increasing? The Baxendell observations give 360 days, and the (later) Rousdon observations give 363. Unless this is part of a fluctuation, *i.e.* is followed by a return to lower than 360, the case is unfavourable to the hypothesis. There are, however, special difficulties in determining a period so near one year.

R Leonis: 312 days. Decreasing with fluctuation (see above).

R Urs. Maj.: 302 days. Decreasing. Baxendell's residuals (*M.N.*, 77, 133) fit periods of 302·5 days from E = 5 to 38. Then 301·9 days (adopted) spans the gap from E = 38 to E = 66, when the observations had been re-started by the B.A.A.: and after that the period is distinctly shorter by two days, as the following corrections show:—

E =	66	67	68	69	70	71	72	73	74
r_1 =	+11	+ 7	+ 3	0	+ 3	− 4	− 3	− 5	− 8
Correction =	− 8	− 6	− 4	− 2	0	+ 2	+ 4	+ 6	+ 8
r_2 =	+ 3	+ 1	− 1	− 2	+ 1	− 2	+ 1	+ 1	0

Thus the period of 300 days fits the B.A.A. observations remarkably well. That the period was distinctly longer in Baxendell's time is supported by Pogson's observations (*Mem. R.A.S.*, 58, xix.), where a period of 302·4 days is independently deduced. The question is somewhat complicated by the occurrence of a sudden jump (see *M.N.*, 76, 480), but this does not reverse the conclusion of a longer period in early days.

S Delphini: 280 days. Decreasing. Cycles II. to IV., 283 days; IV. to X., 280 days; after that, 274 days (*M.N.*, 74, 590).

R Boötis: 224 days. Decreasing? E 12 to 36, 224·03 days, E 36 to 43, 223·16 days (*M.N.*, 74, 458).

Having thus followed Group II. downwards, we will continue in the suggested order of growth through Group I. upwards.

Group I.

T Herculis: 165 days. No change? A uniform period fits all the observations except Baxendell's second series, which is persistently 20 days in error. Is this a case like the "halts" of W Cygni? And are these "halts" a feature of the short periods near the junction of Group II. and Group I.?

R Arietis: 186 days. Increasing slightly. Periods 186·56, 186·82, 187·14 (*M.N.*, 73, 135).

R Persei: 210 days. Increasing with fluctuation. Periods 207·4, 211·2, 209·9 (*M.N.*, 76, 491).

S Urs. Maj.: 226 days. Decreasing? E = − 2 to + 36, 227·5 days. After this 225·0 days. Unless this is part of a fluctuation, to be followed by an increase, the case is unfavourable to the hypothesis.

S Herculis: 305 days. Increasing, with fluctuation (see above).

R Serpentis: 356·8 days. Increasing. The adopted period fits the modern observations (*M.N.*, **77**, 578), but a smaller period of 352·8 days would improve the earlier ones, thus:

E =	31	35	37	38	39	40	
r_1 =	+26	+ 7	− 2	− 8	− 9	−11	days.
Correction =	−23	− 7	+ 1	+ 5	+ 9	+13	,,
r_2 =	+ 3	0	− 1	− 3	0	+ 2	,,

S Orionis: 413 days. Change doubtful (*M.N.*, **76**, 190).

U Canis Min.: 412 days. No secular change, but fluctuation:—424, 412, 400, 412, 424 (*M.N.*, **78**, 506).

U Cygni: 465·3 days. No change (*M.N.*, **74**, 575). By an unfortunate slip U Cygni is entered twice in *M.N.*, **79**, 372 and 378. In the earlier table the name and number should be V Cygni 203847. The other figures are correct.

Summary.

(*a*) Six observers of W Cygni show systematic differences easily allowed for. (§§ 1–6.)

(*b*) The observations of all six show a periodicity ultimately fixed at 129·60 days, with a coefficient of 0·36 magnitude, represented by the variable θ; but there are changes of phase too large to be neglected. (§§ 7–9.)

(*c*) These are chiefly due to a second periodicity in 243 days with coefficient 0·26 magnitude, represented by the variable ϕ. The ratio of periods is closely 15 to 8. (§§ 10–13.)

(*d*) On allowing for this term, the discrepancies in phase mentioned in (*b*) largely disappear, as shown in Table IX.

(*e*) But there remains a considerable "halt" amounting to 48 days, either suddenly or in three parts (§ 14). The variation appears to be arrested for this time and then to resume at the point where it left off.

(*f*) There is also possibly a variation in 1944 days or $5\frac{1}{3}$ years, represented by $\chi = 2\phi - \theta$. But the second and third harmonics of χ are comparable with the first. We may write the whole variation, as in § 23,

$$f(\chi) = \cdot 08 \cos(\chi - 193^\circ) + \cdot 09 \cos(2\chi - 4^\circ) + \cdot 06 \cos(3\chi - 270^\circ),$$

or we may represent it approximately by four straight lines.

(*g*) The total variation is thus represented by

$$6\cdot 10 + 0\cdot 36 \cos(\theta - 336^\circ) + 0\cdot 26 \cos\phi + f(\chi).$$

(*h*) So far as can be seen at present, the whole cycle of changes repeats in 1944 days, for which a diagrammatic ephemeris is given.

But three qualifications are required. Firstly, there is a slow drop in mean magnitudes, which may represent a term of long period: secondly, there is the liability to sudden halt, during which the variation seems to be arrested; thirdly, there is the θ/z term.

The first initial dates for the diagrammatic ephemeris are

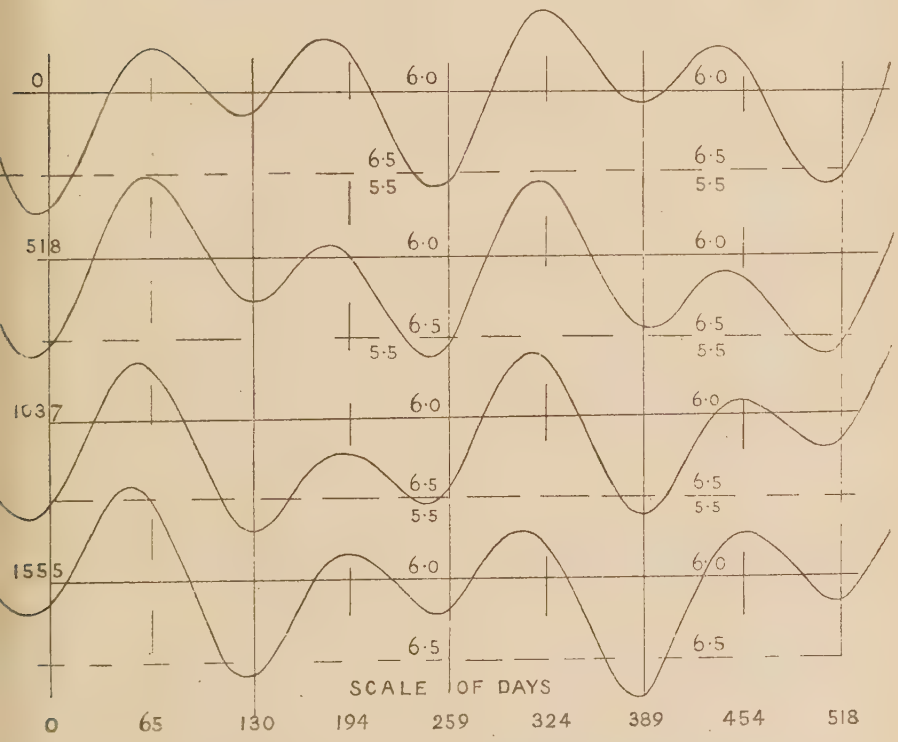
1907 May 8 = J.D. 2417704

and

1912 Sept. 2 = J.D. 2419648

But owing to the halt or halts, of 48 days in all, the third follows at

1918 Feb. 15 = J.D. 2421640.



(*i*) A systematic search for other terms, reveals a number of small ones which are of doubtful reality. But one negative result is of importance. If the θ and ϕ variation occurred in the *brightness* curve of the star, then, on taking logarithms of brightness to convert into magnitudes, we ought to get terms of the form $\cos\theta\cos\phi$ with assignable coefficients. No such terms are found, and the inference seems to be that the variations really arise in terms of magnitude and not of brightness. (§§ 25–38.)

(*k*) A possible explanation of the double variation is that the star is changing from Phillips's Group II. (represented by the ϕ variation, which is waning), to Phillips's Group I. (represented by the θ variation, which is waxing). (§§ 39–40.) Continued observation of this star is therefore of special importance. It is only fitting to add a word of appreciation of the excellent and persistent observing which has made the above discussion possible.

(*l*) The crossing over from Group II. to Group I. suggests as a possible course of evolution a decrease of period from the 400 days of Group II. down to the 200 days; then crossing over to the 150 days of Group I., and increasing up to 400 days again. An examination of the results obtained from the Baxendell stars shows that the periods of Group II. are generally decreasing and of Group I. increasing.

Note added 1919 *December* 15.

By an oversight the series of maxima and minima of W Cygni tabulated in *H.A.*, 55, 236, was not examined until after the above paper had been sent to press. But this is the less regrettable in that an independent check on the suggested ephemeris is thus provided.

The maxima have been compared with a series repeating in 3888 days, which thus allows of taking into account the term in $\theta/2$. It soon appeared that there had been two other jumps similar in character to that already noted; and it has been therefore assumed that they were *exactly* similar, and that the interval between jumps is uniform, as has been found in the case of other stars. We may for simplicity take each jump in two parts (though there is some evidence for three, as above assumed). These may be put for the third jump, above discussed, at

$$E = 23^{\cdot}5 \text{ and } 27^{\cdot}5,$$

or, if we adopt the Harvard numeration of E (to save confusion), we must add 62 units and thus the jumps are at

$$E = 85^{\cdot}5 \text{ and } 89^{\cdot}5.$$

Those for the first jump are taken at

$$E = 10^{\cdot}5 \text{ and } 14^{\cdot}5,$$

and the second, not well defined by the observations themselves, has therefore been taken at

$$E = 48^{\cdot}0 \text{ and } 52^{\cdot}0.$$

We thus subtract from (or add to) the ephemeris constructed by repetition in 3888 days the quantities indicated at the head of each group in the subjoined table. By restoring them (if desired) we get a repeating ephemeris. We then have the following comparison with the ephemeris.

96 *Days subtracted from C.*

E.	C.	O.	O−C.	E.	O.	C.	O−C.
0	9494	9547	+53	6	0269	0303	+34
1	9639	...	...	7	0413	0438	+25
2	9752	9777	+25	8	0529	0528	− 1
3	9898	9892	− 6	9	0669	0666	− 3
4	0010	0043	+33	10	0791	0794	+ 3
5	0156	0161	+ 5				

72 *Days subtracted.*

11	0949	0963	+14	13	1207	1232	+25
12	1081	1073	− 8	14	1345	1321	−24

48 *Days subtracted.*

15	1488	1486	− 2	26	2915		
16	1629	1616	−13	27	3051	3054	+ 3
17	1746	1740	− 6	28	3173	3168	− 5
18	1888	1868	−20	29	3315		
19	2004	1993	−13	30	3430	3448	+18
20	2146	2131	−15	31	3575	3564	−11
21	2263	2262	− 1	32	4605	4629	+24
22	2403	2409	+ 6	33	4993	5025	+32
23	2523	...	...	34	5376	5361	−15
24	2659	2651	− 8	35	5634	5607	− 7
25	2785	2800	+15				

24 *Days subtracted.*

51	6175	6138	−37

0 *Days subtracted.*

53	6459	6486	+27

We may now give in similar form the maxima for the observations discussed in the paper.

E.	O.	C.	O−C.	E.	O.	C.	O−C.
63	7770	7770	0	75	9312	9309	− 3
64	7882	7877	− 5	76	9453	9443	−10
65	8028	8028	0	77	9570	9572	+ 2
66	8141	8156	+15	78	9712	9711	− 1
67	8285	8282	− 3	79	9828	9836	+ 8
68	8401	8385	−16	80	9970	9961	− 9
69	8541	8540	− 1	81	0087	0097	+10
70	8663	8660	− 3	82	0227	0222	− 5
71	8797	8799	+ 2	83	0347	0343	− 4
72	8929	8919	−10	84	0483	0482	− 1
73	9055	9044	−11	85	0609	0609	0
74	9193	9182	−11				

24 Days added.

86	0763	0771	+ 8	88	1021	1035	+14
87	0899	0894	− 5	89	1163	1170	+ 7

48 Days added.

90	1302	1306	+ 4	92	1560	1559	− 1
91	1447	1452	+ 5				

A Method of Determining the Mean Accidental Variations in Daily Rates of a Number of Chronometers. By N. Liapin.

1. It is known that the rates even of well-compensated chronometers (or clocks) are still subject to some irregularities or variations, which serve to characterise their quality provided that they satisfy two important conditions, viz.:

(*a*) They are purely accidental* and

(*b*) oscillate within certain limits.

These conditions being fulfilled, we shall define the square root of quadratic mean of independent differences of a sufficiently great number of daily rates: $u'_1, u'_2, u'_3 \ldots u'_k, u_{k+1}$ of a chronometer as its mean accidental daily *rate-variation*, which we shall denote by $\pm\epsilon$, viz.:

$$\pm\epsilon = \pm\sqrt{\frac{\Sigma(u'_{i+1} - u'_i)^2}{k}} \quad i = 1, 2, 3 \ldots k.$$

$\pm\epsilon$ may be regarded as *quality index* of a chronometer and must therefore be thought of as a constant.† The mean error $\pm\epsilon_1$ introduced by such a rate-variation into a single rate is connected with $\pm\epsilon$ by means of the relation

$$\pm\epsilon = \pm\epsilon_1\sqrt{2} \quad . \quad . \quad . \quad . \quad (1)$$

We shall define $\pm\epsilon_1$ as mean *error of rate-variation.*

2. Let us suppose now that we have N chronometers at our disposal and that there is no possibility of getting their absolute daily corrections and rates in order to obtain their quality-indices. Such cases may occur, for instance, in expedition, field work, or voyages at sea. We are going to establish here the identities which, we believe, may be of interest and use in such cases.

* On criteria for accidentality of errors see author's article, "On a Fundamental Property of Accidental Errors" in *The Observatory*, **41**, 172.

† For chronometers of high quality $\pm\epsilon$ does not exceed $\pm 0^s.20$.

Denoting N chronometers with letters A, B, C, D, ... N, let us suppose that

A_i, B_i, C_i, D_i, ... N_i are their absolute daily corrections for $(k+2)$ days,
a_i, b_i, c_i, d_i, ... n_i are absolute daily rates,
α_i, β_i, γ_i, δ_i, ... ν_i are absolute daily rate-variations.

One can write schematically:

A_1			B_1			C_1			N_1		
A_2	a_1	α_1	B_2	b_1	β_1	C_2	c_1	γ_1	N_2	n_1	ν_1
A_3	a_2		B_3	b_2		C_3	c_2	. . . . .	N_3	n_2	
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.	.	.
A_{k+2}	a_{k+1}	α_k	B_{k+2}	b_{k+1}	β_k	C_{k+2}	c_{k+1}	γ_k	N_{k+2}	n_{k+1}	ν_k

being

$A_{i+1} - A_i = a_i$; $B_{i+1} - B_i = b_i$; $C_{i+1} - C_i = c_i$; . . . . $N_{i+1} - N_i = n_i$; $i = 1, 2, \ldots . (k+1)$
$a_{i+1} - a_i = \alpha_i$; $b_{i+1} - b_i = \beta_i$; $c_{i+1} - c_i = \gamma_i$; . . . . $n_{i+1} - n_i = \nu_i$; $i = 1, 2, \ldots . k$.

Let M be an auxiliary fictitious chronometer whose daily corrections, rates, and rate-variations are as follows:

$$M_i = \frac{A_i + B_i + C_i + \ldots . + N_i}{n}, \qquad i = 1, 2, 3 \ldots . (k+2)$$

$$m_i = \frac{a_i + b_i + c_i + \ldots . + n_i}{n}, \qquad i = 1, 2, 3 \ldots . (k+1)$$

$$\mu_i = \frac{\alpha_i + \beta_i + \gamma_i + \ldots . + \nu_i}{n}, \qquad i = 1, 2, 3 \ldots . k.$$

The number of chronometers being sufficiently great, we may suppose the rate of this fictitious chronometer M to be very nearly perfect, and determine relative rate-variations with regard to M of all N chronometers. We have thus

$$(\mu_i)_A = \mu_i - a_i = \frac{a_i + \beta_i + \gamma_i + \ldots . + \nu_i}{n} - a_i = -\frac{n-1}{n}a_i + \frac{1}{n}\beta_i + \frac{1}{n}\gamma_i + \ldots . + \frac{1}{n}\nu_i$$

$$(\mu_i)_B = \mu_i - \beta_i = \frac{a_i + \beta_i + \gamma_i + \ldots . + \nu_i}{n} - \beta_i = \frac{1}{n}a_i - \frac{n-1}{n}\beta_i + \frac{1}{n}\gamma_i + \ldots . + \frac{1}{n}\nu_i$$

$$(\mu_i)_C = \mu_i - \gamma_i = \frac{a_i + \beta_i + \gamma_i + \ldots . + \nu_i}{n} - \gamma_i = \frac{1}{n}a_i + \frac{1}{n}\beta_i - \frac{n-1}{n}\gamma_i + \ldots . + \frac{1}{n}\nu_i$$

. . . . . .

. . . . . .

$$(\mu_i)_N = \mu_i - \nu_i = \frac{a_i + \beta_i + \gamma_i + \ldots . + \nu_i}{n} - \nu_i = \frac{1}{n}a_i + \frac{1}{n}\beta_i + \frac{1}{n}\gamma_i + \ldots . - \frac{n-1}{n}\nu_i$$

From these equations we obtain easily

$$\sum \frac{(\mu_i)_A^2}{k} = \left(\frac{n-1}{n}\right)^2 \sum \frac{\alpha_i^2}{k} + \frac{1}{n^2}\sum \frac{\beta_i^2}{k} + \frac{1}{n^2}\sum \frac{\gamma_i^2}{k} + \ldots + \frac{1}{n^2}\sum \frac{\nu_i^2}{k}$$

$$\sum \frac{(\mu_i)_B^2}{k} = \frac{1}{n^2}\sum \frac{\alpha_i^2}{k} + \left(\frac{n-1}{n}\right)^2 \sum \frac{\beta_i^2}{k} + \frac{1}{n^2}\sum \frac{\gamma_i^2}{k} + \ldots + \frac{1}{n^2}\sum \frac{\nu_i^2}{k}$$

$$\sum \frac{(\mu_i)_C^2}{k} = \frac{1}{n^2}\sum \frac{\alpha_i^2}{k} + \frac{1}{n^2}\sum \frac{\beta_i^2}{k} + \left(\frac{n-1}{n}\right)^2 \sum \frac{\gamma_i^2}{k} + \ldots + \frac{1}{n^2}\sum \frac{\nu_i^2}{k}$$

. . . . . .

. . . . . .

$$\sum \frac{(\mu_i)_N^2}{k} = \frac{1}{n^2}\sum \frac{\alpha_i^2}{k} + \frac{1}{n^2}\sum \frac{\beta_i^2}{k} + \frac{1}{n^2}\sum \frac{\gamma_i^2}{k} + \ldots + \left(\frac{n-1}{n}\right)^2 \sum \frac{\nu_i^2}{k}$$

probable values of sums of double products being nearly zero.

For the sake of brevity let us put

$$\sum \frac{(\mu_i)_A^2}{k} = \mu_A^2,\ \sum \frac{(\mu_i)_B^2}{k} = \mu_B^2,\ \sum \frac{(\mu_i)_C^2}{k} = \mu_C^2, \ldots \sum \frac{(\mu_i)_N^2}{k} = \mu_N$$

$$\sum \frac{\alpha_i^2}{k} = \alpha^2,\ \sum \frac{\beta_i^2}{k} = \beta^2,\ \sum \frac{\gamma_i^2}{k} = \gamma^2, \ldots \sum \frac{\nu_i^2}{k} = \nu^2,$$

so that the preceding equations will be

$$\mu_A^2 = \left(\frac{n-1}{n}\right)^2 \alpha^2 + \frac{\beta^2}{n^2} + \frac{\gamma^2}{n^2} + \ldots + \frac{\nu^2}{n^2}$$

$$\mu_B^2 = \frac{\alpha^2}{n^2} + \left(\frac{n-1}{n}\right)^2 \beta^2 + \frac{\gamma^2}{n^2} + \ldots + \frac{\nu^2}{n^2}$$

$$\mu_C^2 = \frac{\alpha^2}{n^2} + \frac{\beta^2}{n^2} + \left(\frac{n-1}{n}\right)^2 \gamma^2 + \ldots + \frac{\nu^2}{n^2}$$

. . . . . .

. . . . . .

$$\mu_N^2 = \frac{\alpha^2}{n^2} + \frac{\beta^2}{n^2} + \frac{\gamma^2}{n^2} + \ldots + \left(\frac{n-1}{n}\right)^2 \nu^2.$$

After slight modification we obtain

$$n^2\mu_A^2 = n(n-2)\alpha^2 + (\alpha^2 + \beta^2 + \gamma^2 + \ldots + \nu^2)$$

$$n^2\mu_B^2 = n(n-2)\beta^2 + (\alpha^2 + \beta^2 + \gamma^2 + \ldots + \nu^2)$$

$$n^2\mu_C^2 = n(n-2)\gamma^2 + (\alpha^2 + \beta^2 + \gamma^2 + \ldots + \nu^2)$$

. . . . . .

. . . . . .

$$n^2\mu_N^2 = n(n-2)\nu^2 + (\alpha^2 + \beta^2 + \gamma^2 + \ldots + \nu^2).$$

Adding and putting

$$\mu_A^2 + \mu_B^2 + \mu_C^2 + \ldots + \mu_N^2 = s^2$$
$$\alpha^2 + \beta^2 + \gamma^2 + \ldots + \nu^2 = \sigma^2$$

we have

$$n^2 s^2 = n(n-2)\sigma^2 + n\sigma^2,$$

$$\sigma^2 = \frac{n}{n-1} s^2.$$

We obtain finally the desired identities: *

$$\left.\begin{aligned}
\alpha^2 &= \frac{n}{n-2}\left\{\mu_A^2 - \frac{s^2}{n(n-1)}\right\} \\
\beta^2 &= \frac{n}{n-2}\left\{\mu_B^2 - \frac{s^2}{n(n-1)}\right\} \\
\gamma^2 &= \frac{n}{n-2}\left\{\mu_C^2 - \frac{s^2}{n(n-1)}\right\} \\
&\cdots\cdots\cdots \\
\nu^2 &= \frac{n}{n-2}\left\{\mu_N^2 - \frac{s^2}{n(n-1)}\right\}
\end{aligned}\right\} \quad \ldots \quad (A)$$

which will be so much the more exact as the rate of M is more perfect.

These identities can be easily verified if quality-indices

$$\pm\alpha,\ \pm\beta,\ \pm\gamma,\ \ldots\ \pm\nu$$

are known. If they are to be determined we must recur to the comparison of chronometers.

3. Let us suppose that all chronometers are compared with one of them, say C (comparator), during $(k+2)$ days and all comparisons are reduced to any arbitrary moment, the same for all, according to C. It is easily seen that comparisons give only relative corrections, rates, and rate-variations with regard to C, viz.:

$$A'_i = A_i - C_i;\ B'_i = B_i - C_i;\ C'_i = C_i - C_i = 0;\ \ldots\ N'_i = N_i - C_i$$
$$a'_i = a_i - c_i;\quad b'_i = b_i - c_i;\quad c'_i = c_i - c_i = 0;\quad \ldots\quad n'_i = n_i - c_i$$
$$\alpha'_i = \alpha_i - \gamma_i;\quad \beta'_i = \beta_i - \gamma_i;\quad \gamma'_i = \gamma_i - \gamma_i = 0;\quad \ldots\quad \nu'_i = \nu_i - \gamma_i.$$

Each day of comparison gives immediately

$$(\mu_i)_C = \mu_i - \gamma_i = \frac{\alpha'_i + \beta'_i + 0 + \delta'_i + \ldots + \nu'_i}{n}$$
$$(\mu_i)_A = (\mu_i)_C - \alpha'_i$$
$$(\mu_i)_B = (\mu_i)_C - \beta'_i$$
$$\cdots\cdots\cdots$$
$$(\mu_i)_N = (\mu_i)_C - \nu'_i$$
$$(\mu_i)_C + (\mu_i)_A + (\mu_i)_B + (\mu_i)_D + \ldots + (\mu_i)_N = 0$$

* See for reference author's paper in *Travaux de l'Observatoire de l'Université Impériale d'Odessa*, No. 3, 1915 (in Russian).

and after $(k+2)$ days have elapsed, we can calculate μ_{C}^2, μ_{A}^2, μ_{B}^2, μ_{D}^2, ... μ_{N}^2 as follows:

$$\mu_{\mathrm{C}}^2 = \frac{1}{k}\sum\left\{\frac{\alpha'_i+\beta'_i+0+\delta'_i+\ldots\ldots+\nu'_i}{n}\right\}^2$$

$$\mu_{\mathrm{A}}^2 = \frac{1}{k}\sum\{(\mu_i)_{\mathrm{C}}-\alpha'_i\}^2$$

$$\mu_{\mathrm{B}}^2 = \frac{1}{k}\sum\{(\mu_i)_{\mathrm{C}}-\beta'_i\}^2$$

$$\mu_{\mathrm{D}}^2 = \frac{1}{k}\sum\{(\mu_i)_{\mathrm{C}}-\delta'_i\}^2$$

. . . . .

. . . . . .

$$\mu_{\mathrm{N}}^2 = \frac{1}{k}\sum\{(\mu_i)_{\mathrm{C}}-\nu'_i\}_2 .$$

Substituting these values in (A) we find the required quality-indices. As a test of computation the identity

$$\sigma^2 = \frac{n}{n-1}s^2$$

may be employed.

4. It is important to note that any one of the expressions (A) fails, if

$$\mu_{\mathrm{Q}}^2 < \frac{s^2}{n(n-1)}.$$

This inequality may subsist in three cases; either

(*a*) the chronometer Q is more perfect than the auxiliary chronometer M;

(*b*) there is systematic variation in the rate of the Q chronometer;

(*c*) there exists occasionally an unfavourable combination or accumulation of errors.

In order to overcome this latter inconvenience let us return to the formula

$$\sigma^2 = \frac{n}{n-1}s^2,$$

which may be written in its developed form as follows:

$$\alpha^2+\beta^2+\gamma^2+\ldots\ldots+\nu^2 = \frac{n}{n-1}(\mu_{\mathrm{A}}^2+\mu_{\mathrm{B}}^2+\mu_{\mathrm{C}}^2+\ldots\ldots+\mu_{\mathrm{N}}^2).$$

Equating term to term we may put as closely approximate though not strictly correct,

$$\alpha^2 = \frac{n}{n-1}\mu_A^2$$

$$\beta^2 = \frac{n}{n-1}\mu_B$$

$$\gamma^2 = \frac{n}{n-1}\mu_C^2 \quad . \quad . \quad . \quad . \quad . \quad (B)$$

. . .

. . .

$$\nu^2 = \frac{n}{n-1}\mu_N^2.$$

If there is a sufficiently great number of chronometers, formulæ (A) and (B) give practically the same result, as for $n = \infty$

$$\frac{n}{n-1} = \frac{n}{n-2} = 1 \qquad \frac{s^2}{n(n-1)} = 0,$$

and therefore

$$\alpha^2 = \mu_A^2,\ \beta^2 = \mu_B^2,\ \gamma^2 = \mu_C^2 \ . \ . \ . \ . \ \nu^2 = \mu_N^2 \quad . \quad . \quad (C)$$

Having obtained quality-indices, mean errors of rate-variation and relative weights of chronometers may be found as follows:

$$\pm\alpha_1 = \pm\frac{\alpha}{\sqrt{2}},\ \pm\beta_1 = \pm\frac{\beta}{\sqrt{2}},\ \pm\gamma_1 = \pm\frac{\gamma}{\sqrt{2}} \ . \ . \ . \ . \ \pm\nu_1 = \pm\frac{\nu}{\sqrt{2}}$$

$$W_A = \frac{h}{\alpha^2},\ W_B = \frac{h}{\beta^2},\ W_C = \frac{h}{\gamma^2} \ . \ . \ . \ . \ W_N = \frac{h}{\nu^2},$$

h being an undetermined constant.

University Observatory,
Odessa, Russia:
1917 *December* 25.

Theoretical Establishment of W. Struve's Empirical Formulæ for Errors of Extrapolated and Interpolated Corrections of Clocks or Chronometers due to the Accidental Variations in their Daily Rates. By N. Liapin.

1. The influence of accidental rate-variation on extrapolated and interpolated corrections of clocks has been studied by W. Struve,* who from numerous observations empirically established the laws to which that influence is subject. Later P. Smysloff† fully confirmed these laws for chronometers also. We are going to show that W. Struve's empirical formulæ can also be established theoretically, using the theory of errors only.

First let us calculate the mean error E_e, which produces the effect of rate-variation on extrapolated correction. Let u' be the daily rate with which the extrapolation is made, say for τ days in advance. The mean error of rate-variation is $\pm\epsilon_1$, so that we have $u' \pm \epsilon_1$. The whole change of chronometer correction during τ days is

$$u'_1 + u'_2 + u'_3 + \ldots + u'_\tau = u' + u' + u' + \ldots + u' \ (\tau \text{ times}),$$

where each u'_i may be imagined as quite independent of the others, bearing the mean error $\pm\epsilon'$. The errors of both sides being the same, we obtain, according to the theory of errors, for E_e two expressions:

$$E_e = \pm\epsilon'\sqrt{\tau} = \pm\epsilon_1\tau. \quad . \quad . \quad . \quad (2)$$

W. Struve (*loc. cit.*, p. 100) has shown that formula $\pm\epsilon_1\tau$ fits the observations well. By the nature of our problem either ϵ' or ϵ_1 must be constant, as quality-index. From (2) we conclude that the choice of that constant is a very important matter indeed.‡ It is easily seen that the relations

$$\pm\epsilon'\sqrt{\tau} = \pm\epsilon_1\tau \text{ and } \pm\epsilon'\sqrt{\tau'} = \pm\epsilon_1\tau'$$

are incompatible; ϵ' and ϵ_1, therefore, cannot both remain invariable and either ϵ' or ϵ_1 must be regarded as a function of the time.§

2. In order to obtain the mean error E_i of an interpolated correction u_i let us eliminate u' from the following equations:

$$u = u_1 + u'\tau \pm E_e \qquad u = u_2 - u'\tau' \pm E'_e.$$

* *Expédition chronométrique entre Poulkova et Altona*, St. Petersbourg, 1844.

† *Repsold's Circle, Chronometers, Chronometrical Expedition in 1859*, p. 70, St. Petersburg, 1863 (in Russian).

‡ It will be important to note, that instead of ϵ_1 W. Struve adopts $\mathfrak{e} = \pm 0{\cdot}674\epsilon$, *i.e.* probable accidental daily rate-variation, which, lying within limits of ϵ_1, nevertheless is very nearly equal to $\epsilon_1 = \pm 0{\cdot}707\epsilon$ (see § 1 of the preceding paper).

§ On this subject see B. Wanach's detailed paper, "Über die Genauigkeit interpolierten und extrapolierten Uhrcorrectionen und Gänge," *A.N.*, 190, No. 4546.

E_e and E'_e being mean errors of extrapolation for τ and τ' days respectively, so that we obtain

$$u_i = \left(\frac{u_1\tau'}{\tau+\tau'} \pm \frac{E_e\tau'}{\tau+\tau'}\right) + \left(\frac{u_2\tau}{\tau+\tau'} \pm \frac{E'_e\tau}{\tau+\tau'}\right).$$

According to the theory of errors we have in general

$$E_i = \pm\sqrt{\frac{E_e^2\tau'^2 + E'^2_e\tau^2}{(\tau+\tau')^2}}.$$

Let us assume ϵ' as a constant, and put

$$E_e = \pm\epsilon'\sqrt{\tau} \qquad E'_e = \pm\epsilon'\sqrt{\tau'} \quad . \quad . \quad . \quad (3)$$

we obtain, therefore,*

$$E'_i = \pm\epsilon'\sqrt{\frac{\tau\tau'}{\tau+\tau'}} \quad . \quad . \quad . \quad . \quad (3')$$

Now let us assume ϵ_1 as a constant, and put

$$E_e = \pm\epsilon_1\tau \qquad E'_e = \pm\epsilon_1\tau' \quad . \quad . \quad . \quad (4)$$

Making use of equation (1) of the preceding paper, we find

$$E'_i = \pm\epsilon\frac{\tau\tau'}{\tau+\tau'} \quad . \quad . \quad . \quad . \quad (4')$$

That is precisely W. Struve's empirical formula.† Although both formulæ (3′) and (4′) are correct, we cannot put $E'_i = E^1_i$ because (3) and (4) are incompatible, as we have just seen. This condition subsists only if $\tau' = \tau$.

We are thus led to the conclusion that assumption (3), though quite correct theoretically, is to be rejected, as the imagined subdivision of the time-intervals $u'\tau$ and $u'\tau'$ into τ and τ' independent parts must be regarded as purely artificial.

University Observatory,
Odessa, Russia:
1917 *December* 25.

* W. Struve, *loc. cit.*, p. 98; see also W. Chauvenet, *Manual of Spherical Practical Astronomy*, 1868, 2, 83.

† W. Struve, *loc. cit.*, p. 101; W. Chauvenet, *loc. cit.*, p. 84. W. Struve gives probable error in putting æ instead of ϵ.

A Proposal to construct New Tables for finding Position-Lines at Sea.

By Herbert Bell, M.A., B.Sc.

(*Communicated by the Astronomer Royal.*)

In the usual polar spherical triangle PZσ, where P is the elevated pole, Z the zenith, and σ the star, we have the relation

$$\cos h = \sin a / \cos \delta \cos \phi - \tan \delta \tan \phi \quad . \quad . \quad (1)$$

δ being the declination, ϕ the latitude, h the hour angle, and a the altitude. A single observation gives values for a and δ so that (1) becomes a relation connecting ϕ and h. We may convert h

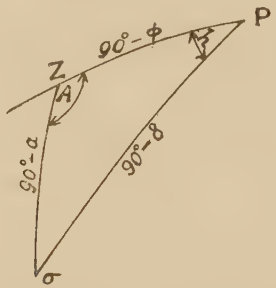


Fig. 1.

into longitude λ by means of the Greenwich mean time (G.M.T.) and the right ascensions. In fact

$$\lambda \text{ (east)} = h \text{ (west)} + (\text{R.A.}\sigma - \text{R.A.M.S.}) - \text{G.M.T.} \quad . \quad (2)$$

by which (1) becomes the (ϕ, λ) locus (Sumner circle) on the earth's surface of the ship's possible positions. On a chart this circle becomes approximately a straight line over short distances, and in particular, if we have a Mercator chart, there is the additional property that directions are correctly reproduced.

The navigator's problem of finding the position-line is to find and draw in a portion (straight) of this locus near the ship's approximately known position. It is plain that a wrong determination of the altitude results in a Sumner circle concentric with the correct one but having a radius in error (in arc) by the error in altitude. That is to say the position-line is thereby translated, on the sphere, parallel to itself a number of miles equal to the altitude error in minutes. Or analytically from (1)

$$-\sin h \, dh = \cos a \, da / \cos \delta \cos \phi$$

h being the position on the ϕ line where the position line cuts it.

i.e. $\sin h \cos \delta \, dh$ or $\sin A \cos a \, dh = -\cos a \sec \phi \, da$.

Now $\sin A \, dh$ is seen to be the perpendicular distance dp between the new and the old position lines, so that

$$dp = -da \sec \phi \quad . \quad . \quad . \quad . \quad (3)$$

giving the rule that the perpendicular distance in miles equals the error in altitude in miles measured on the side of the Mercator chart (sec ϕ). The distortion produced by mapping does not appreciably destroy the pure translation property.

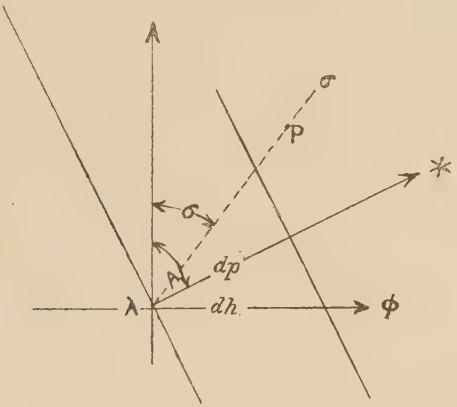


FIG. 2.

It is now proposed to show that if suitable tables were constructed we could considerably simplify the drawing in of this line.

Method A.

Suppose we are between latitudes 20° and 21° north and observe at a certain time the (corrected) altitude of the sun's centre to be $a = 36°\ 44'{\cdot}7$. We may proceed as follows:—Assume $\phi = 20°$, $a = 36°$ the nearest even integral number, and look up δ as usual in the Almanac. The tables, of which part of a page is here given, give us at a single entry both h and the azimuth A. Now convert h by (2) into longitude and draw, through the point λ thus determined on the $\phi = 20°$ line, a line making an angle A,

SPECIMEN PAGE—METHOD A.

Latitude 20°. Declination of Contrary Name.

Alt. 32°.	Δ	Az.	Alt. 34°.	Δ	Az.	Alt. 36°.	Δ	Az.	Alt. 38°.	Δ	Az.	Alt. 40°.
h m s		°	m s		°	m s		°	m s		°	
3 40 54	1·85	104	32 5	1·90	106	23 13	1·97	107	14 18	2·03	108	...
39 3	1·92	106	30 11	1·98	107	21 15	2·05	108	12 16	2·13	109	...
37 8	1·98	107	28 12	2·05	108	19 12	2·12	109	10 8	2·20	111	...
35 9	2·03	108	29 9	2·12	110	17 5	2·20	111	7 56	2·30	112	...
33 7	2·12	110	24 2	2·20	111	14 53	2·28	112	5 38	2·40	114	...
31 0		111	21 50		112	12 36		114	3 14		115	...
...		...	...		...	...		...	...		...	...

with the elevated (north) pole. This points towards the sun, *i.e.* the centre of the Sumner circle. A line perpendicular to this through λ would be the position line on the assumption $a = 36°$, but since $a = 36° + 44'{\cdot}7$ we have finally to draw a parallel line, *i.e.* perpendicular to the drawn line 44·7 miles *nearer* the sun, and the work is completed. We may check our work, and this is not usually possible by other schemes, by working from the 21° latitude line instead, and verifying that the same line is obtained.

As an example let us take a "sight" which is worked out in the Admiralty *Manual of Navigation* (1914), p. 132, by the usual method (slightly modified).

On March 7th, 1914, about 9 a.m. (S.M.T. nearly), in estimated position lat. 19° 45′ N., long. 160° 24′ E., when the deck watch was slow on G.M.T. 2h. 17m. 27s., instrumental error +1′ 30″, height of eye 50 feet, the following observation was taken :—

D.W.	8h. 02m. 36s.	Obs. alt. ⊙	36° 35′ 10″	Decl.	5° 42′·7 S.	
			1′ 30″			
S.M.T.	21h. 00 March 6.		36° 36′·7	Eq.T.	11m. 29s. + to A.T.	
Long.	10 42 36 (E.)	Corr.	+8′·0			
G.D.	10 17 March 6.	a =	36° 44′·7			

From specimen table with

D.W.	8h. 02m. 36s.	Lat. = 20° N.,	Alt. = 36° + 44′·7,	δ = 5° 42′·7 S.	
Slow	2 17 27			Az. = N. 114 E.	
G.M.T.	10 20 03	H.A. =	3h. 13m. 06s.	E.	
			20 46 54	W.	
		Eq.T.	11 29		
			20 58 23		
		G.M.T.	10 20 03		
		Long.	10 38 20	E.	= 159° 35′·0 E.

Draw in the azimuth line through $\phi = 20°$, $\lambda = 159° 35'{\cdot}0$ as shown (full line) and perpendicular to it the required position-line 44·7 miles *towards* the sun.

It will be seen that by means of such tables we obtain the position-line :—

(i) *Without* an estimated (dead reckoning) longitude.
(ii) *Without* separate solution for azimuth from special tables or charts.
(iii) *Without* rules about sizes of angles as in Aquino's tables.
(iv) *With* only *one* entry instead of *three* in Aquino's and *seven* in the present Admiralty method.
(v) *With* an accuracy as great as can be used on sailing charts.

The total size of the tables is not large, and for most sailors who remain within fairly narrow ranges of latitude it would be convenient to divide the tables up into distinct latitude sections.

It may be remarked that a further slight saving of time could be effected if the hour angles written under the different altitude columns had been modified for refraction. If instead of the

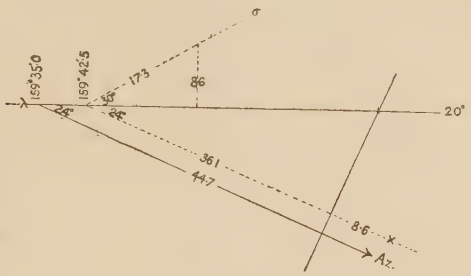


Fig. 3.

correct altitude, say $a = 30°$, we had used as argument $a' = 30°$ uncorrected for refraction, *i.e.* $a = 29°\ 58'\ 20''$ corrected, we should have obtained a slightly different hour-angle set of solutions, with the result that our "sights" would never need correcting for refraction. Since, however, there are various small corrections, dip, parallax, etc., usually grouped along with refraction, in a simple table it is perhaps not worth while adopting this modification.

Method B.

Let us next enquire what amount of translation the position-line suffers owing to slightly erroneous values of the declinations. Taking a and ϕ constant in (1) so as to determine the new point of intersection with a constant ϕ line, we have

$$\begin{aligned} -\sin h\, dh &= (\sin a \sin \delta/\cos^2 \delta \cos \phi - \sec^2 \delta \tan \phi)\, d\delta \\ &= (\sin a \sin \delta - \sin \phi)\, d\delta/\cos^2 \delta \cos \phi \\ &= -\cos a \cos \delta \cos \sigma\, d\delta/\cos^2 \delta \cos \phi \end{aligned}$$

where σ is the parallactic angle.

$$\therefore \sin h \cos \delta\, dh, \text{ i.e. } \sin A \cos a\, dh = \cos a \cos \sigma \sec \phi\, d\delta.$$

As before sin A dh is the perpendicular distance dp' of translation and sec ϕ is the chart scale factor

$$\therefore dp' = \cos \sigma \times d\delta \text{ (measured on chart)} \quad . \quad . \quad (4)$$

we have, therefore, the construction:—Lay off a second line (dotted in fig. 2) making the parallactic angle σ with the elevated pole. Take a length $\lambda P = d\delta$ upon it measured on the side of the chart. The perpendicular distance from P to $\lambda\phi$ is $d\delta \cos \sigma$, the

required translation,* and is away from the sun along the azimuth line. The total correction due to errors in altitude and declination are by (3) and (4)

$$dp = da - \cos\sigma\, d\delta$$

towards the sun. Through the point thus determined upon the azimuth line we finally draw in the true line. To employ this method we need the parallactic angles, and the second specimen page meets this requirement, where the parallactic angles have

SPECIMEN PAGE—METHOD B.

Latitude 20°. *Declination of Contrary Name.*

Dec.	Alt. 32°.	Az.	Par.	Alt. 34°.	Az.	Par.	Alt. 36°.	Az.	Par.	Alt. 38°.	Az.	Par.	Alt. 40°.
°	h m s	°	°	m s	°	°	m s	°	°	m s	°	°	
0	3 42 41	103	66	33 56	104	66	25 7	105	65	16 16	107	64	...
2	39 3	106	65	30 11	107	64	21 15	108	63	12 16	109	63	...
4	35 9	108	63	26 9	110	63	17 5	111	62	7 56	112	61	...
6	31 0	111	62	21 50	112	61	12 36	114	60	3 14	115	59	...
8	26 33	114	60	17 12	115	59	7 45	117	58	58 10	118	57	...
10	21 47	116	59	12 13	118	57	2 30	120	56	52 38	121	55	...
12	16 39	119	57	6 49	121	56	56 49	123	54	46 37	124	52	...
14	...	...	...	...	...	...	...	...	...	...	...	...	...

replaced the previous tabular distances, now no longer necessary since there is no interpolation. We notice that we can now dispense with half the declination entries, *i.e.* halve the tables, since all measurements are now along radii vectores and the possible large h differences along ϕ lines for small hour angles are eliminated. We proceed to solve the former problem by this method.

S.M.T. 21h. 00m. March 6 Obs. alt. ⊙ 36° 35′ 10″ Decl. 5° 42′·7 S.
Long. 10 42 36 1′ 30″ Eq.T. 11m. 29s. + to A.T.
G.D 10 17 March 6 36° 36′·7
Corr. +8′·0
$a = 36°\ 44'·7$
D.W. 8h. 02m. 36 Lat. 20° Alt. 36° + 44′·7. δ 6° − 17′·3 S.
Slow 2 17 27 Solutions H.A. = 3h. 12m. 36s. E. Az. = 114° Par. = 60°
G.M.T. 10 20 03 = 20 47 24 W.
Eq.T. 11 29
S.M.T. 20 58 53
G.M.T. 10 20 03
Long. 10 38 50 E. = 159° 42′·5 E.

* A better method is to take cos $\sigma\, d\delta$ directly from the traverse tables. I owe this suggestion to Instructor-Commander A. E. Monro, R.N.

Lay off azimuth and parallactic angles as shown (dotted), measure off 17·3 miles on parallactic line and subtract the perpendicular distance to 20° from the 44·7 miles laid off along the azimuth line. We get the same position-line as before.

Method B seems preferable to Method A in that the tables are only half the size and there is no interpolation, usually a source of error for the ordinary sailor, although he has here an extra line to draw in. Furthermore, when extreme cases arise (small hour angles) we can always so choose the integral parts of altitude and declination that the corrections oppose each other, and therefore the λ point on the ϕ line is nearer the position-line.

Great Circle Sailing.—It will readily be seen that with either of these tables we can solve the problem of great circle sailing over a considerable range much more easily than with tables now in use. For if in fig. 1 P be the pole of the earth, Z the ship's position, and σ the point to be reached (port), *i.e.* ϕ is latitude as before, h is difference in longitude of Z and σ expressed in time, and δ is latitude of port, then the tables give directly the required course A, measured from the elevated pole and taking the complement of the tabular altitude (zenith distance) the approximate distance in miles. For example, if as in example above we are in lat. 20° 15′ N., 160° 39′ E. and require to reach a point 149° W. long., 3° 10′ S. lat., we enter the tables at latitude 20° and following the row for declination 3° contrary name we run through successive pages until the hour-angle is 161° + 149° = 310° = 50° = 3h. 20m. Then the required course as azimuth is seen to be N. 109° E., and the distance (zenith distance) is 90° − 36° = 3240 miles.

All problems of star finding, compass correcting, etc., are, of course, solved at once.

On the Possible Use of Sidereal Time in Navigation.

Navigators interconvert longitude and hour-angle by means of the equation Long. (east) = hour-angle (west) + right-ascension observed body − right-ascension mean sun − Greenwich mean time, or, in symbols,

$$\lambda = h + \text{R.A. } x - \text{R.A.M.S.} - \text{G.M.T.} \quad . \quad . \quad (1)$$

This equation gives rise in practice to two methods of procedure. First when x is the sun. This case is so common that special tables are made out for R.A. sun − R.A.M.S., called "equation of time," and (1) becomes

$$\lambda = h + \text{Eq.T.} - \text{G.M.T.} \quad . \quad . \quad . \quad (2)$$

Secondly when x is the moon or star. Here (1) is used in full and in the case of the moon both R.A. moon and R.A.M.S. have to be taken out—troublesome operations, as they both involve interpola-

tions. The result is that the average observer dislikes moon and star "sights" as they require a slightly different formula from (2) which he has by rote.

Now if we notice that R.A.M.S. + G.M.T. is Greenwich sidereal time (G.S.T.) we can re-write (1) as

$$\lambda = h + \text{R.A.} \odot - \text{G.S.T.} \quad . \quad . \quad . \quad (3)$$

Hence if sights were taken against G.S.T. instead of G.M.T. (3) would apply to all observations with the further advantage that in the case of the moon and stars there is *one interpolation less*, while for sun sights R.A. sun is as simple as Eq.T.

Now G.M.T. and G.S.T. differ by only about four minutes a day, a "rate" easily produced on existing chronometers by slight modifications. Ultimately, too, perhaps wireless stations could be persuaded to signal sidereal time, but meanwhile a simple table could tell us at what sidereal time to expect time signals.

G.M.T. would still be necessary on board, since declinations are given in terms of it. However, a very rough chronometer or a pocket watch would be accurate enough for this purpose. Later, perhaps, declinations might be tabulated in sidereal time.

The Five Aberrations of Lens-Systems. (Second Paper.)
By A. E. Conrady.

I. The Extended Petzval Theorem.

In my first paper on this subject * I insisted very strongly on the new aspect of the Petzval theorem which resulted from the discussion of the results of the simple analysis. I showed that the curved surface defined by the theorem is the natural reference-surface for the location of the images of extra-axial object-points, inasmuch as the two focal lines due to astigmatism were found to lie always on the same side of the Petzval-curve, and the focal line due to tangential rays (*i.e.* those proceeding in the plane of the optical axis) always at a distance from the Petzval-curve exactly three times as large as that of the focal line due to sagittal rays proceeding in a plane at right angles to the plane containing the principal optical axis.

The presence of ordinary spherical aberration does not affect this law at all, for it results also from the theory that the spherical aberration of the oblique pencils is exactly equal to that of an axial pencil of the same aperture: therefore the sharpest foci of all points in the field would be shifted by an equal amount towards or away from the lens-system and would therefore be referred to an equidistant of the Petzval-surface. As in a practically useful lens-system all the residual aberrations must necessarily be small,

* *M.N.*, 79, 60.

the difference in curvature between the Petzval-surface and a close equidistant would be negligible.

If coma is present the case is even simpler, for the simple coma-circle, which marks the point of greatest geometrical concentration of the light, is located exactly *in* the Petzval-surface, so that coma by itself does not shift the latter in any way.

Matters become rather more involved if coma and spherical aberration are present simultaneously: the simple coma-circles corresponding to successive concentric zones of the clear aperture then lie at the distances from the Petzval-surface resulting from the spherical aberration. Taken in their entirety the coma-circles cover the surface of an oblique circular cone with its apex at the point where the principal ray from the centre of the clear aperture cuts the Petzval-surface, and the point of greatest concentration of the light is not easily estimated with accuracy, although it is not difficult to see that the variation of the curvature of the image cannot possibly be considerable.

The extended Petzval theorem which was foreshadowed in the first paper has in the meantime been extensively tested by direct computations of a great variety of lens-systems such as telescope-objectives, eyepieces, and photographic objectives, and has, without exception, been fully confirmed to an even higher degree than could be expected, seeing that the whole theory only takes the primary aberrations into account and neglects the effects of the higher aberrations.

The conclusion arrived at is that the curvature of the sagittal field may be safely derived from the extended theorem without separate calculations in all cases where the Petzval-curvature corresponds to a radius of the order of the focal length of the system (with all ordinary glasses and lens-types the radius of curvature ranges from ·75 to 1·6 times the focal length) provided no angle of incidence or obliquity exceeds about 20°.

The most convenient form of the simple Petzval theorem is

$$(a) \quad \cdot \quad \cdot \quad \cdot \quad \frac{1}{R'_i} = \frac{1}{R_i} - \sum \frac{N-1}{N}\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

R_i being the radius of curvature of the surface on which the object-points lie, R'_i the radius of curvature of the corresponding image-surface in the absence of astigmatism, and the sum being taken for all the simple lenses of which the lens-system is built up, N being the refractive index, r_1 the radius of the first surface and r_2 the radius of the second surface of each lens, the numbering of the radii expressing the order in which the light reaches the surfaces. The thickness and separation of the constituent lenses does not affect R'_i at all. The extension of the theorem resulting from the discussion in my previous paper may be expressed mathematically by the equation

$$(b) \quad \cdot \quad \cdot \quad \cdot \quad \frac{1}{R'_t} - \frac{1}{R'_i} = 3\left(\frac{1}{R'_s} - \frac{1}{R'_i}\right) \quad \cdot$$

if R'_t stands for the radius of the surface on which the tangential focal lines lie and R'_s for the radius of the corresponding sagittal image-surface.

As a single equation expressing the extended Petzval theorem I propose the form resulting from the combination of (*a*) and (*b*), namely,

$$\text{I.} \quad \frac{3}{R'_s} = \frac{1}{R'_t} + \frac{2}{R_i} - 2\sum^{N-1}_{N}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

In the vast majority of practical applications the original object will be assumed to be plane, and the term $2/R_i$ then vanishes.

The equation I. expresses a fundamental law of image-formation by oblique pencils within the accuracy of the primary aberrations, that is, discrepancies between I. and the directly computed R'_s can only be due to the presence of sensible amounts of aberrations of the 5th or higher orders.

Disregard of the important qualification just stated explains the much-discussed nonfulfilment of the Petzval-condition in all existing so-called anastigmatic photographic objectives. It is quite true that in all these systems the Petzval-sum leads to a value of R'_t of the order of five to twenty times the equivalent focal length, always in the sense of a round field, instead of the infinite value which would render possible a flat field free from astigmatism. But it is also true, and easily verified either by calculation or by a sufficiently searching test of the actual system (as, for instance, by the Hartmann method), that no such thing as a true anastigmat exists or is even reasonably possible. All these systems have curvature of the field or, more usually, the corresponding amount of overcorrected astigmatism in exact agreement with my equation I. in the central part of the field, say, up to 10° or 15° from the optical axis. At and beyond this point the 5th and higher order aberrations become sensible and rapidly grow in magnitude with increasing obliquity: in the successful designs they work in opposition to the tendency of the 3rd order aberrations and lead to freedom from astigmatism and curvature of field in some well-chosen zone, usually at a distance of about 30° from the optical axis, beyond which zone the higher order aberrations gain the upper hand, reversing the sense of both astigmatism and curvature of field. The conditions thus exactly correspond to those discussed in detail in my recent paper on Star-discs (*Monthly Notices*, 79, 575) with reference to the spherical aberration terms of successive orders, and the computer's skill is called upon to effect such a balance that the residuals do not reach an objectionable magnitude in any part of the intended field. The so-called anastigmats therefore fully deserve their name only with reference to one narrow zone of the field which usually lies beyond the range of obliquity within which the Petzval theorem can be legitimately applied.

II. *Influence of the Position of the Diaphragm.*

The general theory of the third order aberrations given in the first paper readily admits of an extremely useful generalisation. In developing the theory it was assumed that there exists somewhere a diaphragm, concentric with the optical axis of the lens-system, by which the course of the rays actually admitted is regulated and restricted. As is well known, the effects produced by a lens-system may be profoundly affected not only by changing the clear aperture of the diaphragm, but also by shifting the latter along the optical axis. I shall now show how the effect of shifting the diaphragm axially may be readily determined and discussed when the constants of the five aberrations have once been determined for some particular position of the diaphragm. Obviously we then know the exact course of every ray which can pass from

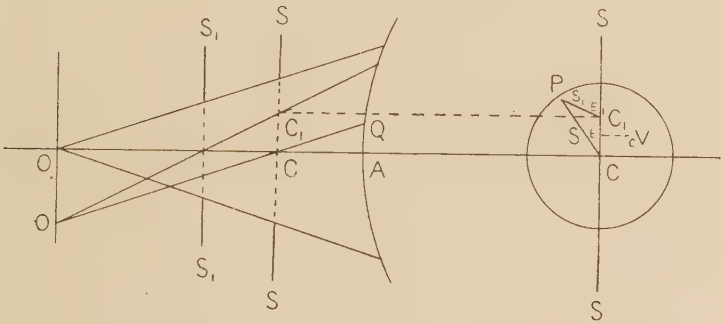


Fig. 1.

Fig. 2.

any point of the clear aperture of the lens, for we only have to put the proper values of V and of S into the equations for the horizontal and vertical displacements of the emerging ray in order to obtain the desired result. Movements of the diaphragm (even those at right angles to the optical axis) cannot therefore bring into action any rays not included in the original solution when a sufficiently large value of S is used: all that can be required is therefore to *define* the rays admitted by any given opening of the displaced diaphragm. This common-sense conclusion is easily put into mathematical form.

Referring to fig. 1, which represents part of fig. 1 of the previous paper, it is obvious that if we replace the diaphragm SS by another S_1S_1 at a different distance from the lens-system, the new diaphragm may be dimensioned so as to admit exactly the same axial pencil: but an oblique pencil limited by S_1S_1 will pass eccentrically through SS, and this eccentricity will clearly be proportional to the distance of the extra-axial object-point O_1 from the optical axis: and as this distance is, within terms of the third order, proportional to the angle V, we can put the distance

from the centre of SS at which the central ray from S_1S_1 passes through the original diaphragm equal to cV where c is a constant depending solely on the distance between the original and the new diaphragm. If we now take an axial view of the original diaphragm, we have (fig. 2) the point of penetration P of any ray defined with reference to the centre of SS by its distance from that centre, $CP = S$ and the angle E. With reference to the eccentrically displaced centre C_1 of the pencil admitted by S_1S_1, we have the new co-ordinates $C_1P = S_1$ and angle E_1, and we have the relations between S and E and S_I and E_1 respectively.

$$(c) \quad \begin{cases} S \sin E = S_1 \sin E_1 \\ S \cos E = S_1 \cos E_1 + c.V \\ \qquad S^2 = S^2{}_1 + c^2V^2 + 2cVS_1 \cos E_1. \end{cases}$$

In the previous paper we found the following values for the components of the aberrational displacement of any ray from the final focus of the succession of thin radial pencils:

$$(d) \quad \begin{cases} \text{Horiz. comp.} = m_1S^3 \cos E + m_2S^2V(2 + \cos 2E) + 3m_3SV^2 \\ \qquad \cos E + m_5V^3 \\ \text{Vert. comp.} = m_1S^3 \sin E + m_2S^2V \sin 2E + m_3SV^2 \sin E.* \end{cases}$$

If we introduce into these equations the equivalents given by (*c*), which is most easily effected by first putting $2 + \cos 2E = 1 + 2 \cos {}^2E$ and $\sin 2E = 2 \sin E \cos E$ and eventually reintroducing $\cos 2E_1$ and $\sin 2E_1$, we arrive at the transformed equations

II. Horizontal component =	Vertical component =
$= m_1S^3{}_1 \cos E_1$	$= m_1S^3{}_1 \sin E_1$
$+ (m_2 + c.m_1)S^2{}_1V(2 + \cos 2E_1)$	$+ (m_2 + cm_1)S^2{}_1V \sin 2E_1$
$+ 3(m_3 + 2cm_2 + c^2m_1)S_1V^2 \cos E_1$	$+ (m_3 + 2cm_2 + c^2m_1)S_1V^2 \sin E_1$
$+ (m_5 + 3cm_3 + 3c^2m_2 + c^3m_1)V^3$	

Remembering that c is proportional to the lateral displacement of the effective oblique pencils, we see that the first terms in each series, indicating spherical aberration, are quite unaffected by diaphragm-shifts. The coma represented by the second terms changes solely in direct proportion to c and to the spherical aberration. The astigmatism (third terms) changes in simple proportion to c and the original amount of coma and only in a secondary manner with the spherical aberration, the coefficient c^2 being very small for small diaphragm shifts.

Finally we find the last term, distortion, growing in direct proportion to c and the original amount of astigmatism, secondarily

* The sequence of the terms and the numbering of their coefficients have been changed to bring them into accordance with Seidel's five aberrations.

with the coma and only in a tertiary manner with the spherical aberration.

A highly interesting feature is that Seidel's fourth aberration —the Petzval-condition—finds no place in this sequence: it evidently holds the distinctive position as an invariant of any given system which is so clearly brought out in my way of deducing the oblique aberrations. Another interesting and important fact emerges: that with the exception just stated, Seidel's sequence of the aberrations receives another logical justification, and the only one which is not subject to any exceptions or reservations. Shifts of the diaphragm result primarily in a change of each aberration proportional to the magnitude of that immediately preceding it in the sequence. Seidel based his sequence purely on consideration of the sharpness of the image: as spherical aberration affects the whole field of a system equally, it must be removed first in order to secure good definition in the optical axis. Coma comes next because it grows in direct proportion with the angle of the field and thus becomes sensible quite close to the optical axis, and so for the following aberration. But this is only true when high resolution and definition in the central part of the field are the decisive requirements. When accurate measurements, especially on photographs, are intended, then coma undoubtedly is the most serious aberration, as the only one leading to unsymmetrical deformation of the extra-axial images. And in lens-systems required to cover a field of large angular extent, astigmatism and curvature of field must take first place.

Yet another interesting result, again justifying the sequence adopted by Seidel, is obtained when we ask for what value of c the separate aberrations in II. attain a maximum or minimum value. Taking the distortion, differentiation of its coefficient with reference to c gives the exact coefficient of the astigmatic term, thus indicating a maximum or minimum when the astigmatism is zero. The second differential coefficient of the distortion-factor is equal to the coefficient of the coma-term: therefore the distortion will be a maximum if the astigmatism is zero and the coma negative, a minimum if astigmatism is zero and coma positive. A precisely similar result follows for the astigmatic term. This reaches extreme values for that value of c which makes the coma zero, and this extreme value is a maximum if the spherical aberration is negative ("overcorrected") and a minimum if the spherical aberration is positive or "undercorrected."

Many of the above results, taken by themselves, are of great value to the practical designer and even to the user of lenses, but their value is immensely increased by combining with the deductions from II. those resulting from a general theorem concerning curvature of field and applicable to thin lenses which, if known at all, certainly has not found the appreciation which it deserves. If narrow fans of sagittal (s) or tangential (t) rays are refracted at a surface (fig. 3) which at the point of incidence has a uniform curvature to radius r and which separates a medium of refractive

index N from another of index N′, and if I is the angle of incidence of the central ray of each fan, I′ the corresponding angle of refraction, then the distances of conjugate points from the point of

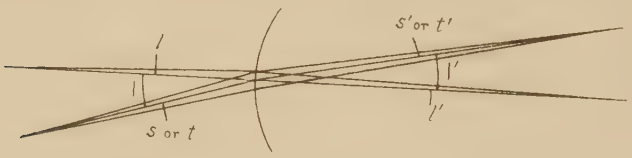


FIG. 3.

incidence, s and t before refraction, s' and t' after refraction, are determined by the well-known equations

$$(e) \quad \begin{cases} \dfrac{N'}{s'} = \dfrac{N' \cos I' - N \cos I}{r} + \dfrac{N}{s} \\ \dfrac{N' \cos^2 I'}{t'} = \dfrac{N' \cos I' - N \cos I}{r} + \dfrac{N \cos^2 I}{t} \end{cases}.$$

For a normally incident pencil ($I = I' = 0$) these yield the corresponding paraxial equation between conjugate distances l and l'

$$\frac{N'}{l'} = \frac{N' - N}{r} + \frac{N}{l}.$$

In the special case when the two astigmatic fans of rays pass *centrally* through a *thin* lens (fig. 4) with radii of curvature r_1 and r_2 the angle I_1 of incidence at the first surface becomes equal to the angle I'_2 of emergence from the second surface, and the angles I'_1

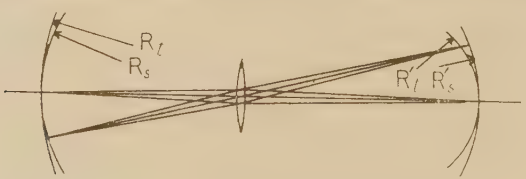


FIG. 4.

and I_2 within the lens are also equal. This leads to such far-reaching simplifications that it is quite easy to determine the curvature of the resulting image-surfaces to radius R'_s for sagittal, R'_t for tangential pencils by neglecting terms of the fourth and higher orders, with the remarkable result, closely analogous to the Petzval-equation

$$(f) \quad \begin{cases} \dfrac{1}{R'_s} = -\dfrac{N+1}{N.f} + \dfrac{1}{R_s} \\ \dfrac{1}{R'_t} = -\dfrac{3N+1}{N.f} + \dfrac{1}{R_t} \end{cases}$$

R_s and R_t being the radii of the surfaces on which the sagittal and tangential object-points (more correctly focal lines) lie, f the focal length of the thin lens and N its refractive index.

The curvature of the two astigmatic image-surfaces is thus shown to be quite independent of the shape of the thin lens (whether biconvex, plano-convex, or meniscus) and of the conjugate distances of object and image: the curvature is a true constant. Its magnitude is surprising, for whilst the image of a flat object produced by a thin lens of suitable form and with a diaphragm at the proper distance to secure freedom from astigmatism has the radius of curvature resulting from the Petzval-equation $R'_i = -N.f$ or for the usual crown-index of 1·5 $R'_i = -1{\cdot}5f$, equation (f) gives for a similar lens passed *centrally* by the oblique pencils the very short radii of curvature of the image-surfaces:

$$R'_s = -.6f \text{ and } R'_t = -.273f.$$

The equations are of such a form that we can immediately write down the result for a combination of any number of thin lenses in contact, when passed centrally by the oblique pencils, namely,

$$\text{III.} \quad \frac{1}{R'_s} = \frac{1}{R_s} - \sum \frac{N+1}{Nf} \text{ and } \frac{1}{R'_t} = \frac{1}{R_t} - \sum \frac{3N+1}{Nf}.$$

These equations give very similar ratios between the R'_s and R'_t and the combined focal length of the system as that found above for a simple lens, but in achromatic combinations of the usual glasses, in which the flint has a higher index than the crown, the curvatures of the image-surfaces are even worse and may be roughly estimated as ranging around $R'_s = -.58f$ and $R'_t = -.27f$.

A very noteworthy result is obtained by forming the difference of the two equations III. which gives

$$\frac{1}{R'_s} - \frac{1}{R'_t} = \frac{1}{R_s} - \frac{1}{R_t} + 2\sum \frac{1}{f}.$$

This difference obviously measures the astigmatism directly, and this is shown to be independent even of the refractive index of the lenses: it is a pure function of the focal length and of nothing else.

The proof of these remarkable equations depends upon the assumption that the lenses are, strictly speaking, infinitely thin. But direct theoretical discussion as well as empirical calculations agree in showing that any ordinary thickness of the lens or lenses affects the result but slightly and that even when the thicknesses are so considerable as to cause a decided departure from the curvature deduced from III. for pencils passing as nearly centrally as is possible, it requires only a very moderate decentration of the oblique pencils by a diaphragm suitably placed fairly close to the lens or lens-combination to restore values in reasonable agreement with III. For this reason equations III. form a very important addition to the methods of arriving at preliminary estimates of the

performance of lenses. When taken in conjunction with II. they supply ready answers to a vast number of lens problems, of which only a few of the most interesting and important ones can be mentioned in this paper.

(1) Ordinary telescope-objectives—that is, those composed of two fairly thin lenses placed close to each other—necessarily have the curvature of the image and heavy undercorrected astigmatism determined by III. when used at full aperture or with a diaphragm close to the object-glass. When corrected as well as is possible—that is, for freedom from coma *and* spherical aberration—m_1 and m_2 in II. will be zero: consequently the astigmatism is constant and cannot be altered by shifting the diaphragm. The latter operation would only alter the distortion, and as this is nearly zero when the pencils pass centrally through the lens, the effect of a distant diaphragm would be worse than that of a diaphragm close to the lens. The case for astrographic object-glasses properly computed and made in accordance with the specification of the Congress is thus a hopeless one as far as flattening of the field and removal of astigmatism are concerned.

(2) A very large proportion of existing astronomical object-glasses do not fulfil the sine-condition: there is then a sensible residue of coma, and m_2 is nearly always positive whilst m_1 is very nearly zero. In object-glasses of this type, when stopping down is resorted to, a very great improvement of the field will result by placing the diaphragm at a considerable distance from the lens, for this will leave the coma quite unaffected but will diminish or even overcorrect the astigmatism. With the very prevalent type consisting of a nearly equiconvex crown followed by a nearly concavoplane flint the diaphragm should be placed at about 1/3 the focal length from the lens towards the focus.

(3) A very remarkable property of spherically overcorrected thin lens-systems results from II. and III. taken together. As was shown in the preliminary discussion of II., the astigmatism of any spherically overcorrected lens-system reaches a maximum when the coma is brought to zero by a suitably placed diaphragm. From this it follows that it is impossible to free a spherically overcorrected fairly thin lens from coma without incurring the penalty of heavy astigmatism and curvature of field in the undercorrected sense. For without a diaphragm, with all the pencils passing centrally, the overcorrected lens will have the heavy curvature of field resulting from III. If under these conditions it is free from coma, no more need be said. But if there is coma, and a more or less distant diaphragm is used to remove it, the astigmatism will be increased to its maximum value possible in that lens, and the curvature of field will necessarily be aggravated.

(4) The fact, established on a sound theoretical basis by II. and III., that reasonably thin lenses or combinations of lenses if spherically corrected or overcorrected can have their field flattened, even by a distant diaphragm, only if a considerable amount of coma is allowed and that they will then give

highly objectionable unsymmetrical images of extra-axial object-points, supplies the simple and sufficient explanation why lenses of that type required for a large field, chiefly photographic "landscape" lenses and achromatic "positive" eyepieces, almost invariably are found to be spherically undercorrected. We proved that in the presence of spherical undercorrection freedom from the objectionable coma coincides with the *minimum* value of astigmatism and curvature of field and thus makes a good compromise easily attainable.

(5) Referring to the preceding case, we can also draw conclusions as to the form the lens (or combination) must take to secure the desired result. Assuming that the m-values in II. have been calculated for central passage of the oblique pencils, and so represent the case to which III. applies, it follows that to remove the coma by a diaphragm we must have $c = -m_2/m_1$, and as m_1 must be positive, c and m_2 must have opposite signs. Under the usual optical sign-conventions to which our equations are subject, a positive c corresponds to a position of the diaphragm to the left of the lens, and a positive m_2 to a higher magnification by paraxial rays than that given by marginal rays, the focal plane of the lens, if convex or positive, being always assumed to lie to the right of it. In the case of a landscape-lens producing real images of distant objects a positive m_2 will require (as every lens-designer knows) a lens turning a strong convexity towards the distant objects and very probably a concavity towards the focus: *vice versa* for a negative m_2. As c and m_2 must have opposite signs, this supplies the general proof of the well-known empirical fact that single photographic lenses must be strongly convex on the side farthest from the diaphragm and nearly always present a concavity to the diaphragm. Closely built positive eyepieces are brought under this rule by considering the Ramsden-disc (through which all the emerging oblique pencils pass) as the diaphragm and reversing the course of the rays, *i.e.* looking upon the eyepiece as capable of forming in the focal plane a reasonably flat image of very distant objects situated on the side of the observer.

(6) Equations II. also supply the definite proof of a theorem which has hitherto been merely accepted because no exception to it is known: that thin single lenses which depend for the correction of the oblique aberrations on a diaphragm at some distance from the lens always display a considerable amount of distortion. Again taking the central passage of the oblique pencils as our starting-point, we shall have the fixed positive value of m_3, the astigmatic coefficient, which resulted from equation III. According to the required aperture and sharpness of definition we shall also be able to fix the maximum permissible value of the spherical undercorrection m_1. The problem, then, is to decide upon such a form of the lens that with a suitable value of c the coma will be corrected and the astigmatic factor will have a desirable value, say A. The coma condition gives $m_2 = -c.m_1$. Putting this into the analytical expression for the astigmatic factor, we next find

$c^2 = (m_3 - A)/m_1$. If we then substitute the values found into the distortion-coefficient we find,

$$\text{Distortion-coeff.} = m_5 + c(m_3 + 2A).$$

As m_5 is the distortion-coefficient when the oblique pencils pass centrally through the lens it is practically zero, as has already been stated. Hence the distortion with the diaphragm in position is determined by the second term of the last equation. This could only become zero if $A = -\frac{1}{2}m_3$. So large an amount of over-corrected astigmatism is out of the question, and therefore there will always be considerable distortion with an otherwise well-corrected lens of this type. Further discussion on the lines adopted in the previous section shows that there will be barrel-distortion with the usual position of the diaphragm in front of the lens, pincushion-distortion with a diaphragm behind the lens, in exact accordance with experience. The theoretical possibility of removing the distortion by admitting heavy astigmatic over-correction appears to be a new, though hardly a useful, fact.

(7) As a last and decidedly important deduction from II. and III., we will discuss two widely separated reasonably thin lenses which are required to be free from coma and spherical aberration and to have a favourable correction of astigmatism. Under these conditions the combination, taken as a whole, will be independent of the position and aperture of a diaphragm, as was pointed out in the preliminary discussion of II. We can therefore imagine a diaphragm placed into any position which simplifies the discussion without detracting from the validity of the latter. We will take a combination of the type of the Petzval portrait-lens as an example (fig. 5) and place our imaginary diaphragm into the back lens. The latter alone would then have the great curvature of field and under-corrected astigmatism defined by III., quite regardless of its form. To secure a better state of affairs for the complete combination the front lens must therefore be made to supply a correspondingly overcorrected pencil. Now the front lens is working through a diaphragm behind it, or with a negative value of c in equation II. With centrally passing pencils it will have the inevitable round field, but the astigmatic coefficient in II. shows that with a negative value of c we can reduce or even overpower the large positive value of m_3, first and foremost by giving to the front lens a large positive value of m_2 and to a lesser, secondary, extent by also giving it a negative m_1. We saw in a previous section that a positive value of m_2 calls for a lens having a strong convexity towards the distant objects and a concavity towards the image: hence this form is necessary in the front lens of binary combinations of the type dealt with in this section. In order to compensate the positive coma of the front lens, the back lens must have a corresponding amount of negative coma, and this decides its form, which must present considerable convexity towards the focus.

This simple theory of binary systems supplies the explanation of a very undesirable characteristic of all microscope-objectives

in which the highest possible correction of spherical aberration simultaneously in two colours has been the designer's chief aim. They all have a very round field. The explanation in the light of the above theory is quite simple: looked at in the reverse position, producing a minified image of a distant object, a microscope objective assumes the function of a portrait lens of very large relative aperture. The forms of the component lenses which were shown to be indispensable for the correction of astigmatism and curvature of field in such a combination are the worst possible from the point of view of high spherical correction and especially of fulfilment of the Gauss-condition. Flatness of field and high definition at large apertures are thus proved to be incompatible conditions, which cannot be fulfilled simultaneously.

I have only indicated briefly and qualitatively some of the important results which can be readily derived from the three fundamental equations for oblique pencils I., II., and III., taken either separately or conjointly. It will be readily accepted that their value is greatly increased when they are made the basis of

Fig. 5.

algebraical treatment or of numerical calculations. As was pointed out in the first paper, the four aberration-constants can be quickly and accurately determined by tracing only 5 rays indicated by numbers 1 to 5 in figure 6, in the plane of the optical axis; trigonometrically through any centred system, no matter how complicated it may be. The emerging rays are traced to their intersection with the Petzval-curve, taken as a circle of radius R'_1 and laid through the paraxial image-point, and then give the horizontal component of the aberrational displacement directly. All these calculations call only for the use of the simple ray-tracing formulæ given in my paper on chromatic correction (*Monthly Notices*, **64**, 182). For reasonably thin lenses or systems an even simpler method is often sufficient: we merely trace a paraxial and a marginal ray for the axial image-point, these give the spherical aberration directly and the coma by means of the sine-condition. The astigmatism and curvature of field are then found with fair approximation by III. and the distortion may be put down as practically zero for the centrally passing oblique pencils to which these simple computing methods apply. From the analytical purist's point of view these trigonometrical computations for rays at finite angles are reprehensible, because the results will be affected by the higher aberrations which are not taken into account in the equations I. to III. by which the results are to be discussed and

interpreted. The practical lens-designer, however, takes the opposite view: if the rays have been well chosen with reference to the intended aperture and field of the lens-system, the exact ray-tracing will lead to a very favourable distribution of the residual higher aberrations and to a great reduction of their maximum value, in the way which I discussed in detail with reference to axial image-points in the later parts of my paper on Star-discs (*Monthly Notices*, **79**, 575), and if the modifications resulting from the discussion of the first results are of moderate extent, this advantage will be largely preserved and will render the second approximation a closer one than could be expected if the higher aberrations had been totally neglected.

In conclusion I will point out that the trigonometrical tracing of the 5 rays specified above shows the possibility of an even vaster generalisation of the general theory of oblique pencils than that dealt with in this paper. We arrived at equations II. by noticing that the equations (*d*) for the Seidel-aberrations included every

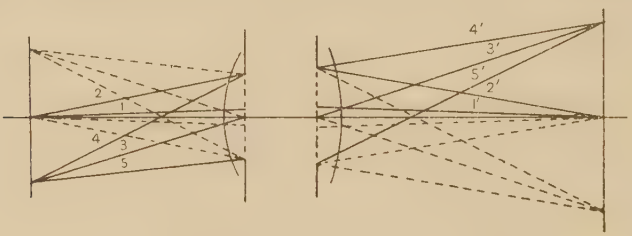


FIG. 6.

ray passing from the adopted object-surface through the lens-system. Evidently they really include every ray that can pass through the lens-system from *any* point in the object space. But in the usual analytical treatment we cannot make immediate use of this conclusion because it gives us only one point of each emerging ray with the precision to 3rd order terms required for the discussion of the aberrations, namely, its intersection with the Petzval-surface laid through the adopted axial image-point. The *direction* of the emerging rays required only to be known with 1st order accuracy for the purpose of determining the Seidel-aberrations for a given pair of conjugates. Obviously the determination of the direction of the emerging rays with 3rd order accuracy will be secured by the precise location of a second point of each ray at a finite distance from the image-surface. The tracing of the 5 rays in fig. 6 readily supplies the necessary data for such a second point if one small addition is made: for reasons of symmetry the calculation of the 5 rays shown in full lines is equally valid for the 5 rays shown in dotted lines which pass through points at the same distance on the opposite side of the optical axis and at the opposite slope. It is only necessary to reverse the sign of all angles to secure the data of these dotted rays. This gives us three entering

rays intersecting respectively in the upper and lower edge of the entrance-pupil and the corresponding emerging rays locate the corresponding points of the exit-pupil. The entering ray aiming at the centre of the entrance-pupil similarly locates the centre of the exit-pupil for the corresponding emerging ray at a finite angle. If, as the addition alluded to, we calculate also the paraxial image of the centre of the entrance-pupil we obtain the spherical aberration-constant for its image the exit-pupil, and we then have all necessary data for the determination of the Seidel-aberrations in the exit-pupil from which the required second point on any emerging ray is obtainable with the required accuracy. The wider prospect thus presented has not yet been fully explored, but there seems to be before us a hopeful expectation of further extensions to the knowledge of the properties of oblique pencils and of simplifications in the calculation of complicated systems.

Imperial College of Science and Technology:
1919 *November* 13.

Occultations of Stars by Planets in 1920. By Arthur Burnet.

(*Communicated by Professor H. H. Turner.*)

1. Occultation of No. 4649 Bordeaux (1890), mag. 8, by Venus, 1920 Jan. 4.

Geocentric conjunction in R.A., 1920 Jan. 4, $23^h\ 28^m$ G.M.T.
R.A. of planet and star $15^h\ 58^m\ 41^s{\cdot}71$.
Declination of planet $-18°\ 2'\ 47''{\cdot}0$.
Declination of star $-18°\ 2'\ 49''{\cdot}0$.

The occultation is visible in the eastern parts of North America, and the duration, as seen from Washington, will be about five minutes—$23^h\ 23^m$ to $23^h\ 28^m$ G.M.T.

2. Occultation of 31 Sagittarii, mag. 6·7, by Venus, 1920 Feb. 7.

Geocentric conjunction in R.A., 1920 Feb. 7, $1^h\ 0^m$ G.M.T.
R.A. of planet and star $18^h\ 47^m\ 20^s{\cdot}13$.
Declination of planet $-22°\ 0'\ 45''{\cdot}6$
Declination of star $-22°\ 0'\ 54''{\cdot}6$

The occultation will be visible over a somewhat restricted area in the United States.

3. Occultation of B.D. $+18°$ 2094 (mag. 8·9), by Jupiter, 1920 Feb. 19.

Geocentric conjunction in R.A., 1920 Feb. 19, $19^h\ 14^m$ G.M.T.
R.A. of planet and star $8^h\ 54^m\ 42^s{\cdot}10$.
Declination of planet $+18°\ 21'\ 47''{\cdot}4$.
Declination of star $+18°\ 22'\ 1''{\cdot}3$.

The occultation is visible throughout America (North and South), and the approximate times are—beginning $18^h\ 28^m$ G.M.T., ending $20^h\ 24^m$ G.M.T.

4. Occultation of No. 1460 Cape (1900), mag. 7·3, by Saturn, 1920 March 14.

Geocentric conjunction in R.A., 1920 March 14, $8^h\ 5^m$ G.M.T.

R.A. of planet and star $10^h\ 38^m\ 6^s{\cdot}78$.

Declination of planet $+10^\circ\ 46'\ 14''{\cdot}4$.

Declination of star $+10^\circ\ 46'\ 10''{\cdot}7$.

The assumed place of the star is based upon the Cape Catalogue (1900), and an annual proper motion of $-^s{\cdot}005$ in R.A. and $-0''{\cdot}02$ in dec. has been applied. This has been derived from a comparison of the following catalogues:—Lalande, Weisse-Bessel (1825), Washington (1860), Leipzig (1875), and Cape (1900).

The position of the planet is the mean of those given in the *Nautical Almanac* and the *Connaissance des Temps*, which differ by $^s{\cdot}09$ in R.A. and $0''{\cdot}3$ in dec. On these data the particulars of the occultation as seen from Greenwich are as follows:—

	h	m	
Disappearance of star behind ring . .	6	35	1920 Mar. 14 G.M.T.
,, ,, ,, planet . .	7	15	
Reappearance from ring (outer circumference)	8	35	
,, ,, planet	8	45	

It will be noted that, according to the above, the reappearance of the star from behind the ring is invisible, being obscured by the planet. A very slight error in the adopted position of either the planet or star will have a very appreciable effect upon the circumstances of the occultation.

The phenomenon should be visible throughout the whole of the Eastern hemisphere, including also Western Australia.

5. Occultation of B.D. $+18^\circ$ 2103, mag. 8·8, by Jupiter, 1920 May 24.

Geocentric conjunction in R.A., 1920 May 24, $13^h\ 16^m$ G.M.T.

R.A. of planet and star $8^h\ 58^m\ 8^s{\cdot}78$.

Declination of planet $+18^\circ\ 2'\ 59''{\cdot}2$.

Declination of star $+18^\circ\ 3'\ 11''{\cdot}0$.

The occultation will be visible in the eastern part of North America, beginning at $12^h\ 28^m$ G.M.T. and ending at $14^h\ 3^m$ G.M.T.

6. Occultation of B.D. $+17^\circ$ 2028, mag. 9·0 by Jupiter, 1920 June 7.

Geocentric conjunction in R.A., 1920 June 7, $19^h\ 46^m$ G.M.T.

Declination of planet $+17^\circ\ 26'\ 4''{\cdot}5$.

Declination of star $+17^\circ\ 26'\ 17''{\cdot}4$.

An occultation lasting from 19^h G.M.T. to 20^h G.M.T. should be visible from Eastern Australia and New Zealand.

7. Close approach of B.D. -11° 3726, mag. 7·9, to Mars, 1920 April 4.

Geocentric conjunction in R.A., 1920 April 4, $19^h\ 30^m$ G.M.T.

R.A. of planet and star $14^h\ 18^m\ 29^s{\cdot}84$.

Declination of planet $-11^\circ\ 42'\ 2''{\cdot}6$.

Declination of star $-11^\circ\ 41'\ 42''{\cdot}0$.

No occultation is therefore possible, but a very close approach might be witnessed in South America.

8. Close approach of B.D. + 18° 2056, mag. 8·9, to Neptune, 1920 May 25.

Geocentric conjunction in R.A., 1920 May 25, $3^h\ 25^m$ G.M.T.

R.A. of planet and star $8^h\ 46^m\ 7^s{\cdot}91$.

Declination of planet + 17° 59′ 8″·9.

Declination of star + 17° 59′ 10″·6.

The position of the star is reduced from the Catalogue d'Abbadia (1900), but is of course subject to some uncertainty.

University Observatory, Oxford:
1919 *December* 12.